Chapter 4
Complexity as a Discourse on School Mathematics Reform

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Abstract This writing begins with a brief introduction of complexity thinking, coupled to a survey of some of the disparate ways that it has been taken up within mathematics education. That review is embedded in a report on a teaching experiment that was developed around the topic of exponentiation, and that report is in turn used to highlight three elements that may be critical to school mathematics reform. Firstly, complexity is viewed in curricular terms for how it might affect the content of school mathematics. Secondly, complexity is presented as a discourse on learning, which might influence how topics and experiences are formatted for students. Thirdly, complexity is interpreted as a source of pragmatic advice for those tasked with working in the complex space of teaching mathematics.

Keywords Complexity thinking • School mathematics • Mathematics curriculum

One of the most common criticisms of contemporary school mathematics is that its contents are out of step with the times. The curriculum, it is argued, comprises many facts and skills that have become all but useless, while it ignores a host of concepts and competencies that have emerged as indispensible. Often the problem is attributed to a system that is prone to accumulation and that cannot jettison its history. Programs of study have thus become not-always-coherent mixes of topics drawn from ancient traditions, skills imagined necessary for a citizen of the modern (read: industry-based, consumption-driven) world, necessary preparations for postsecondary study, and ragtag collections of other topics that were seen to add some pragmatic value at one time or another over the past few centuries – all carried along by a momentum of habit and familiarity. Somewhat ironically, a domain that has not been particularly influential in these evolutions is mathematics itself. As a result, few current curricula have any substantial content that is reflective of developments in mathematics over the past few centuries.

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L. Jao, N. Radakovic (eds.), *Transdisciplinarity in Mathematics Education*,
DOI 10.1007/978-3-319-63624-5_4

Oriented by a deep concern for this situation, I am currently involved in a longitudinal investigation of "changing the culture of mathematics teaching at the school level." Through this design-based inquiry, a group of university researchers has teamed with the staff of a school in a 7-year commitment to work together in transforming how mathematics is seen and engaged.

The project has three foci, distilled from preliminary discussions with the teacher-participants:

- Mathematics curriculum – e.g., what mathematics is important to teach? Is that the same as what is in the curriculum? Where did that curriculum come from?
- Individual understanding – e.g., how does understanding of a concept develop? Is there a "best" way to structure/sequence teaching to support robust conceptual development? Are individuals' understandings necessarily unique, or is there a way of nudging learners to "true" interpretations of concepts?
- Social process – e.g., how do groups support/frustrate the development of individual understanding? How does individual understanding support/frustrate the work of groups?

At first blush, the range of topics represented in these clusters of questions may seem to be so broad as to disable inquiry. In truth, even as one of the principal researchers, I was at first taken aback with the full range of concerns raised by the research partners. However, while these three clusters of questions might seem on the surface to be focused on disparate matters, "inside" them there is a uniting theme: complexity.

More precisely, each of these clusters of issues concerns a category of emergent phenomena. That is, each points to a form or agent that obeys an evolutionary dynamic and that arises in and transforms through the interactions of other forms and agents. That realization helped to shift the principal focus from the three clusters of questions above to a single unifying theme. In the process, as is reported below, a space was opened both to move toward productive and pragmatic responses to the questions posed and to make meaningful strides toward the grander intention of the project.

What Is "Complexity" within Mathematics Education?

Before getting into some of the specifics of those developments, it is important to situate the intended meaning of *complexity*. Unfortunately, there is no unified or straightforward definition of the word. Indeed, most commentaries on complexity research begin with the observation that there is no singular meaning of complexity, principally because researchers tend to define it in terms of their particular research foci. One thus finds quite focused-and-technical definitions in such fields as mathematics and software engineering, more-indistinct-but-operational meanings in chemistry and biology, and quite flexible interpretations in the social sciences (cf. Mitchell, 2009).

Within mathematics education, the range of interpretations of complexity is almost as divergent as it is across all academic discourses. This variety can in part be attributed to the way that mathematics education straddles two very different domains. On one side, mathematics offers precise definitions and strategies. On the other side, education cannot afford such precision, as it sits at the nexus of disciplinary knowledge, social engineering, and other cultural enterprises. Conceptions of complexity among mathematics education researchers thus vary from the precise to the vague, depending on how and where the notion is taken up.

However, diverse interpretations do collect around a few key qualities. In particular, *complex* systems adapt and are thus distinguishable from *complicated* (i.e., mechanical) systems that may be composed of many interacting components and which can be described and predicted using laws of classical physics. A complex system comprises many interacting agents – and those agents, in turn, may comprise many interacting subagents – presenting the possibility of global behaviors that are rooted in but not reducible to the actions or qualities of the constituting agents. In other words, a complex system is better described by using Darwinian dynamics than Newtonian mechanics.

Complexity research only cohered as a discernible movement in the physical and information sciences in the middle of that twentieth century, with the social sciences and humanities joining in its development in more recent decades. To a much lesser (but noticeably accelerating) extent, complex systems research has been embraced by educationists whose interests extend across such levels of phenomena as genomics, neurological process, subjective understanding, interpersonal dynamics, mathematical modeling, cultural evolution, and global ecology. As discussed elsewhere (Davis & Simmt, 2014, 2016), these topics can be seen across three strands of interest among mathematics education researchers – namely:

- Regarding the contents of curriculum, complexity as a disciplinary discourse – i.e., as a digitally enabled, modeling-based branch of mathematics
- Regarding beliefs on learning, complexity as a theoretical discourse – i.e., as the study of learning systems, affording insight into the structures of knowledge domains, the social dynamics of knowledge production, and the intricacies of individual sense-making
- Regarding pedagogical strategies, complexity as a pragmatic discourse – i.e., as a means to nurture emergent possibility, with advice on how to design tasks, structure interactions, etc.

For the most part, to my reading, researchers in mathematics education have tended to treat these issues singularly. That is perhaps not surprising, since each represents a significant departure from entrenched, commonsense beliefs. However, as I attempt to illustrate in the example I turn to presently, there may be great transformative potential in treating these considerations as necessary simultaneities.

Importantly, the resonance between these three strands of interest among mathematics education researchers and the three foci of the project (mentioned earlier) are not accidental. Engaging with teachers about such matters is, I believe, integral to bringing possibilities afforded by complexity thinking to the realities experienced by teachers. This thought has oriented much of my own research efforts over the past several years, particularly around efforts to co-design and co-teach units of study with teachers in our design-based research study. To that end, in the following account, I endeavor to highlight how complexity can serve, simultaneously, as a theory of curriculum, learning, and pedagogy.

A Teaching Experiment on Exponentiation

As already noted, for centuries, the basics of school mathematics tend to be construed as addition, subtraction, multiplication, and division. Notably, these operations are "basic" not because they are foundational to mathematics knowledge, but because they were vital to a newly industrialized and market-driven economy a few hundred years ago. It is easy to see why computational competence would be useful to a citizen of that era and to ours as well. If anything, the need has been amplified in our number-dense world. However, it is not clear that these four operations are a sufficient set of basics today, given that some of the most pressing issues – such as population growth, the rise of greenhouse gases, ocean acidification, decline in species diversity, cultural change, increases in debt, and so on – have strongly exponential characters. More descriptively, these sorts of pressing issues are instances of complexity, evidenced in part by their potentials for rapid change and unpredictability.

Understandings and appreciations of the volatility of prediction have become rather commonplace, evidenced in the way the "butterfly effect" has captured the collective imagination. However, while awareness of this popular trope might suggest that complexivist sensibilities have gained traction, it might also indicate limited understanding of the actual mechanisms at work inside complex dynamical systems. The butterfly effect is most often stated in terms of a system's sensitivity to initial conditions, but what really matters is the power of iteration to amplify or dampen. That is, the butterfly effect – like any complex dynamic – only makes sense within a frame of exponentiation.

I mentioned that thought in a social conversation with an eighth-grade teacher in Calgary, and she promptly challenged me to design and teach a brief unit in which exponentiation was treated as a useful interpretive tool rather than a site for symbolic manipulations. The major impetus for the work was thus professional curiosity rather than a predefined research intention. (Appropriate ethical clearances and permissions were secured.) She generously offered a week of lessons, and a few weeks later, I found myself in her regular-stream class of 32 students. Not wanting to interrupt established routines much, I mimicked the teacher's structures of frequent full-group discussion, modulated with small-group work. No individual seatwork and no deliberate homework were assigned during the week. That decision was made for several reasons. Firstly, the brevity of the project made it difficult for me to get to know the students and communicate expectations in ways that made

Table 4.1 An overview of a weeklong unit on exponentiation

Day	Focus	Activities
Monday	Images of exponentiation	Drawing pictures of exponential change Web searches ("exponentiation," "exponential growth," "powers of two," and related terms)
Tuesday	Exponentiation lattice	Collectively assembling a lattice Looking for patterns Contrasts to addition and multiplication lattices
Wednesday	Analogies to other binary operations	Symbolism and vocabulary Noting similarities between addition and multiplication, and extending these to exponentiation
Thursday	Exploring the validity of those analogies	Justifying and questioning Thinking about the structure of mathematics and mathematical ideas
Friday	Consolidation and examples	Other illustrations of exponentiation Instances of exponentiation in the world we inhabit

me confident such emphases would be effective. Secondly, and closely related, a driving intention of the unit was to trouble the conflation of "mathematics" and "computation" – and, to my mind, individual seatwork and homework presented risks of pressing those two constructs together. Thirdly, as a champion of collective sense-making, I am personally much more comfortable in settings where learners have ample opportunity to express their thinking, to challenge one another, and to openly speculate.

The outline of lesson topics for that week is presented in Table 4.1. A more detailed, general overview of the classroom activities has been presented elsewhere (Davis, 2015), and so only summary descriptions are offered here.

The unit's opening task was an invitation to create images of exponential change. Students were instructed on drawing grid-based images of sequential doubling – starting by outlining a single square, then doubling the figure to enclose two squares, and so on, to the limits of their sheets of paper. T-tables were incorporated into the activity to record quantities and make number patterns more apparent, and students were then tasked with creating similar images and tables for bases of 3–9. They were encouraged to do Web searches and together generated a rich range of associated figures that included images of exponential growth/decay and exponential curves.

On the second day, students were asked to compare exponentiation to addition and multiplication. Earlier in the school year, the class had created poster-sized lattices for addition, subtraction, multiplication, and division on *xy*-coordinate grids. On these charts, values on the *x*-axis served, respectively, as augend, subtrahend, multiplier, and dividend; values on the on the *y*-axis as addend, minuend, multiplicand, and divisor; and corresponding positions on the grid as locations for sums, differences, products, and quotients. Figure 4.1 presents small portions of these lattices.

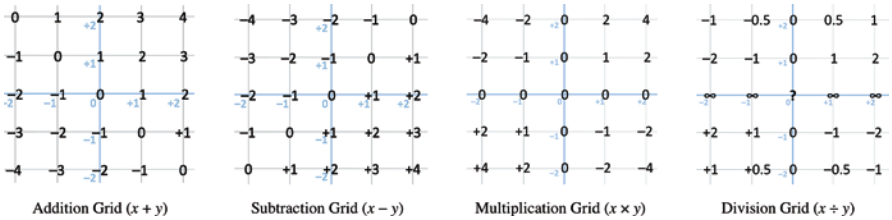


Fig. 4.1 Core portions of the addition, subtraction, multiplication, and division lattices generated earlier in the school year

In the earlier unit, these devices proved to be powerful tools for noticing patterns and, in the process, interpreting identity elements, commutativity, and other concepts and properties. We imagined a chart for exponentiation might serve similar purposes and began the second class with the construction of an exponentiation lattice spanning values of -10 to $+10$ on both axes – that is, covering the range of -10^{-10} to 10^{10}. A core portion of the exponentiation lattice is presented in Fig. 4.2.

The collective analysis of the result began by examining the first quadrant. Students compared its patterns to those in the addition and multiplication lattices, posted nearby. Three observations were immediately noted. First, students remarked on the "steeper and crazy-steeper" increases in values as one moves away from the origin, contrasted with the "flattening" feel of the addition lattice and the "gentler rising" of the multiplication lattice. Second, it was noted that the exponentiation chart "doesn't fold over like adding and multiplying" – that is, whereas the addition and multiplication lattices are symmetric about the line $y = x$, the exponentiation lattice is not. Third, "the diagonal of one table is the 2-row of the next." That is, just as the values along the $y = x$ diagonal of the addition lattice correspond to those of the $y = 2$ row of the multiplication lattice, so the values along the $y = x$ diagonal of the multiplication lattice correspond to those of the $y = 2$ row of the exponentiation lattice. Discussions touched on such topics as commutativity and other symmetries, the mathematics of rapid change, logarithms, imaginary numbers, and mathematical notations (see Davis, 2015, for a more complete account on how discussions of these observations unfolded).

The third session dealt with analogies between exponentiation and the operations of addition and multiplication. Prompted by the problems encountered with x^x the previous day, we began by noting that the symbolism for exponentiation might obscure the relationship to other operations. To highlight similarities to "2 + 3" and "2 × 3," we proposed "2 ↑ 3," which is one of several accepted notations (Cajori, 2007). The resulting set of pairs

$x + x = 2x$.

$x \times x = x^2$.

$x \uparrow x = x^x$.

seemed to satisfy the desire for parallel representations that had emerged the day before.

We set up the day's task with a version of Table 4.2 (below), which was an extension of a chart they had done earlier in the year comparing properties of addition to properties of multiplication. We reminded them of that detail to get things started and then invited suggestions for completing the row labeled "commutative property."

The main point of this activity was to deepen understandings of exponentiation. A second purpose was to support understandings of the relationship among concepts, based on a vital difference between topics studied at elementary and secondary levels. Whereas almost all the concepts encountered at the elementary level can be interpreted in terms of (i.e., are analogical to) objects and actions in the physical world, the analogies for concepts at the secondary level are mostly mathematical objects (see Hofstadter & Sander, 2013). Making analogies, then, is both a mechanism for extending mathematical insight and a window into the structure of mathematics knowledge.

Before setting the students to work on their own, we indicated that they should not worry about the last column, as we had already planned that for the focus of the fourth session. The rest of the class was devoted to filling in blank cells, an effort that began in small groups and that ended in whole-group negotiations of acceptable, parallel phrasings for each entry (see the second row in Table 4.3). Notably, the final three rows of the chart were additions proposed by the students themselves.

The fourth session was devoted to exploring the truth or falsity of the conjectures from the day before. Students worked in small groups and focused on speculations of their choosing. They also made free use of the Internet to help them in their deliberations. Topics in the follow-up discussion included a problem with the speculation on inverse values (i.e., that for every a there is a $\downarrow a$ such that $a \uparrow (\downarrow a) = 1$), because the exponentiation grid suggested $a \uparrow 0 = 1$ (for all $a \neq 0$). If the speculation were true, it would mean that the exponentiative inverse of every number would be 0, which most felt to be nonsensical – in addition to rendering the speculation on "operating on the opposite" similarly troublesome. We elected to leave these discussions unsettled, suggesting that our simple analogies might be misleading. We also suggested that further studies in high school would shed some light on a few of the details – a point that was supported by topics that came up in students' Web searches, including logarithms, imaginary and complex number systems, and tetration.

The final session was devoted to review and consolidation. We framed the session by developing the table presented in Table 4.4, through which we suggested that the geometric image best fitted to addition is the line, to multiplication is a rectangle, and to exponentiation is a fractal. That thought was tied in to a "fractal card" activity (Simmt & Davis, 1998) that the students had undertaken earlier in the school year.

The balance of the lesson was given to conducting searches and looking across instances of exponential growth and decay (e.g., creating fractal cards, population growth, species decline, greenhouse gas increase, technology evolution), framed by Charles and Ray Eames' (1977) film, *Powers of Ten* and Cary and Michael Huang's (2012) interactive Prezi, *The Scale of the Universe*. Exponential growth curves emerged to be a uniting image across these explorations and also proved useful as a

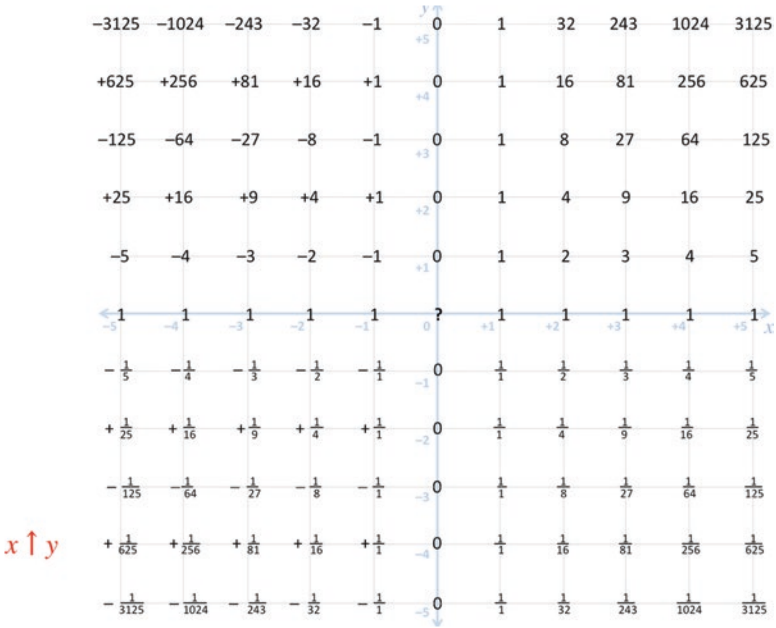


Fig. 4.2 A core portion of the exponentiation lattice

recap on the week as they linked back to the images and grid developed on Monday and Tuesday.

Complexity as a Disciplinary Discourse: Moving from Computation to Modeling

Revisiting the three ways that complexity has been taken up by mathematics education researchers, I would assert that the above teaching episode is an instantiation of those diverse but complementary perspectives on the discourse:

- Complexity as a theory of curriculum – specifically, in this case, an examination of the mathematics of rapid change, which is vital for appreciating the dynamics involved in complex modeling; more generally, approaching mathematics as a means to model experiences and phenomena
- Complexity as a theory of learning – using principles of complexity to interpret individual sense-making, collective knowledge production, and mathematics itself as responsive, adaptive systems that require disequilibration, interactivity, and other conditions of emergence (see Davis & Sumara, 2006)
- Complexity as a theory of pedagogy – used, for example, to inform the distribution of tasks across the collective, to balance redundancy and specialization of agents, and to blend emergent possibilities with preconceived intentions (Davis & Simmt, 2003)

Table 4.2 The blank speculation table

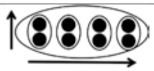
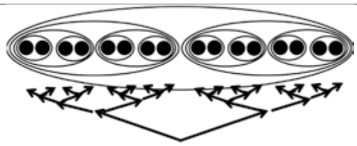
Topic/property	How it looks for addition ($x + y$)	How it looks for multiplication ($x \times y$)	Speculation for exponentiation ($x \uparrow y$)	T/F
Commutative property				
Reverse operation				
Identity element				
Inverse values				

Table 4.3 Conjectures for exponentiation based on analogies to addition and multiplication

Topic/property	How it looks for addition ($x + y$)	How it looks for multiplication ($x \times y$)	Speculation for exponentiation ($x \uparrow y$)	T/F
Commutative property	$a + b = b + a$	$a \times b = b \times a$	$a \uparrow b = b \uparrow a$	False: $2\uparrow 3 \neq 3\uparrow 2$
Reverse operation	Subtraction (−)	Division (÷)	De-exponentiation (↓)	
Identity element	0 … as in $a + 0 = 0 + a = a$	1 … as in $a \times 1 = 1 \times a = a$	1? … since $a \uparrow 1 = a$ … although $1 \uparrow a = 1$	
Inverse values	Additive inverse of a is $0 - a$, or $-a$; $a + (-a) = 0$	Multiplicative inverse of a is $1 \div a$, or $\frac{1}{a}$; $a \times \frac{1}{a} = 1$	Exponentiative inverse of a is $1 \downarrow a$, or $\downarrow a$; $a \uparrow (\downarrow a) = 1$	
Operating on the opposite	Subtraction can be done by adding the [additive] inverse: $a - b = a + (-b)$	Division can be done by multiplying the [multiplicative] inverse: $a \div b = a \times \frac{1}{b}$	De-exponentiation must be doable by exponentiating the [exponentiative] inverse: $a \downarrow b = a \uparrow (\downarrow b)$	
"Next" operation	A repeated addition is a multiplication	A repeated multiplication is an exponentiation	A repeated exponentiation must be a … something	
"Next" set of numbers	When you allow subtraction, you need signed numbers	When you allow division, you need rational numbers	When you allow de-exponentiation, you need another set of numbers	

Each of these points merits considerable elaboration. However, given constraints on space, I focus on the first, with the suggestion that school mathematics might be reconstrued in terms of modeling rather than the currently dominant computation-heavy emphasis. Repeating an assertion made earlier, as I hope is illustrated with the account of the teaching experiment, all three elements must occur simultaneously – and so I acknowledge the artificiality of focusing on the first point. (The citations included in the second and third points provide detailed discussions of those elements.)

Table 4.4 Some geometric analogies to arithmetic operations

Operation	Principal visual metaphors	Common applications/interpretations (using whole number values)
2 + 4		Combining of sets or lengths along 1 dimension Can be consistently represented in linear form
2 × 4		Sets of sets or array/area generated by crossing dimensions Can often be represented as a rectangle
2 ↑ 4		Sets of sets of sets (etc.) or multidimensional form Representable in a fractalesque, recursively generated and/or branching image

This suggestion is, of course, anchored to a conviction that being mathematically competent is about being able to interpret and simulate real-life situations with mathematical constructs. It was in this spirit that exponentiation was studied in the reported classroom episode. While some calculations were involved, computation was always a means to an end. It was a tool within the modeling activity.

To elaborate, a "model" is a representation – a description, an image, a copy – which is intended to highlight vital, defining attributes of some phenomenon. Most often, a model is a simplification, one that is useful as a tool for understanding. A "mathematical model" is thus a description of a phenomenon using mathematical constructs. Examples abound and range from the mundane to the enormously complex. On the more familiar end of the spectrum, every act of counting or measuring is an act of mathematical modeling – that is, of representing a situation in terms of an appropriate number system. At the more complex end of the spectrum, mathematical models are used in the natural sciences (e.g., physics, chemistry, biology, geology, meteorology, astronomy), engineering, and the social sciences (e.g., economics, psychology, political science, sociology) to interpret, explain, and predict phenomena that arise in the interactions of many, many interacting agents.

In this sense, the discipline of mathematics has always been about modeling – although this core emphasis has often been obscured by the computational demands of some models. In particular, prior to rapid and inexpensive computing, the modeling of systems was largely focused on those dynamics that could be studied through differential linear equations. Poincaré was notable among those who examined nonlinear dynamical systems, doing so from a theoretical perspective (Bell, 1937). The computational power of digital technologies in the second half of the twentieth century was necessary for the investigation of dynamical systems began to flourish. Computing power brought about possibility of doing "experimental mathematics" (Borwein & Devlin, 2008) and numerical analysis, triggering a rebirth of the modeling of nonlinear dynamical systems. Importantly, digital computing provided not only a means of computing extremely large data sets and iterating functions through hundreds of thousands of repetitions, it also provided means for converting numerical data to visual representations, enabling the generation of new insights and, consequently, new forms of mathematics (Mitchell, 2009).

It might be tempting to characterize the ever-growing gap between the research mathematics and school mathematics in terms of the contrast between the emphasis on modeling in the former and the emphasis on computation in the latter. That distinction would be unfair, however. Every topic in school mathematics was originally selected for its power to model, and this detail helps to explain the traditional pedagogical emphasis has been on rote application. In the first public schools, learners were being trained not to model, but to apply established mathematical models, and to do so efficiently and effectively. Routinized, repetitive instruction that does not allow for much divergent thinking is arguably the best way to do that.

In other words, schooling's emphasis on computation was a once-fitting educational emphasis, aimed at exploiting mathematics' capacities to model critical elements of one's world. However, circumstances and sensibilities have changed, along with the needs of a mathematically literate citizen. But so too have the affordances of the world in which we live, such as access to data, computational speed, and spatio-visual interfaces. Such evolutions were behind Lesh's (2010) assertion that complexity has emerged as "an important topic to be included in any mathematics curriculum that claims to be preparing students for full participation in a technology-based age of information" (p. 563).

To be clear on the point of this writing, the suggestion is *not* that study of complex systems is new, but that the mathematics of complexity could represent a significant shift from traditional emphases on computation to a new emphasis on (complex) modeling – and, in that shift, possibly nudge school mathematics closer to its parent discipline. As Stewart (1989) has reported, mathematicians have long seen their work in terms of modeling. Just as significantly, they were perfectly aware when they were using linear approximations and other reductions in order to avoid computational intractability. Lecturers and texts followed suit in omitting nonlinear accounts; hence generations of students were exposed to over-simplified, linearized versions of natural phenomena. In other words, non-complex mathematics prevailed in public schools not because it was ideal but because it lent itself to calculations that could be done by hand. The power of digital technologies has not just opened up new vistas of calculation, they have triggered epistemic shifts as they contribute to redefinitions of what counts as possible and what is expressible, and this insight has been engaged by many mathematics education researchers (e.g., English, 2011; Hoyles & Noss, 2008; Moreno-Armella, Hegedus, & Kaput, 2008).

Notable in this the movement toward recasting school mathematics in terms of modeling is the seminal work of Papert (e.g., 1980), particularly his development of the Logo programming language in the late 1970s. The language was designed to be usable by young novices and advanced experts alike. It enabled users to solve problems using a mobile robot, the "Logo turtle," and eventually a simulated turtle on the computer screen. While not intended explicitly for the study of complexity, Logo lent itself to recursive programming and was thus easily used to generate fractal-like images and to explore applications dynamically – opening the door to more complexity-specific topics. To that end, different developers have since offered Logo-based platforms that are explicitly intended to explore complex systems (and other) applications. For example, StarLogo (lead designer, M. Resnick; http://education.mit.edu/starlogo/) and NetLogo (lead designer, U. Wilensky; http://ccl.northwestern.edu/netlogo/). Both platforms were developed in the 1990s and extended Papert's original Logo program by presenting the possibility of multiple, interacting agents (turtles). This feature renders the applications useful for simulating ranges of complex phenomena. Both StarLogo and NetLogo include extensive online libraries of already-programmed simulations of familiar phenomena (e.g., flocking birds, traffic jams, disease spread, and population dynamics) and less-familiar applications in a variety of domains such as economics, biology, physics, chemistry, neurology, and psychology. At the same time, the platforms preserve the simplicity of programming that distinguished the original Logo (e.g., utilizing switches, sliders, choosers, inputs, and other interface elements), making them accessible for even young learners. Other visual programming languages have been developed that are particularly appropriate to students (e.g., Scratch, http://scratch.mit.edu, and ToonTalk, http://www.toontalk.com).

Over the past few decades, hundreds of speculative essays and research reports (see, e.g., http://ccl.northwestern.edu/netlogo/references.shtml) have been published on these and other multi-turtle programs. Regarding matters of potential innovations for school mathematics, in addition to well-developed resources, there have been extensive discussions, and there exists a substantial empirical basis for moving forward on the selection and development of curriculum content that is fitted to themes of complexity. Not surprisingly, then, with the ready access to computational and imaging technologies in most school classrooms, some (e.g., Jacobson & Wilensky 2006) have advocated for the inclusion of such topics as computer-based modeling and simulation languages, including networked collaborative simulations (see Kaput Center for Research and Innovation in STEM Education, http://www.kaputcenter.umassd.edu). In this vein, complexity is understood as a digitally enabled, modeling-based branch of mathematics that opens spaces (particularly in secondary and tertiary education) for new themes such as recursive functions, fractal geometry, and modeling of complex phenomena with mathematical tools such as iteration, cobwebbing, and phase diagrams.

The shift in sensibility from linearity to complexity is more important than the development of the computational competencies necessary for modeling. The very role of mathematics in one's life is transformed through this shift in curriculum emphasis. As Lesh (2010) described, "whereas the entire traditional K–14 mathematics curriculum can be characterized as a step-by-step line of march toward the study of single, solvable, differentiable functions, the world beyond schools contains scarcely a few situations of single actor–single outcome variety" (p. 564). Extending this thought, Lesh highlighted that questions and topics in complexity and data management are not only made more accessible in K-14 settings through digital technologies, but current tools have also made it possible to render some key principles comprehensible to young learners in manners that complement traditional curriculum emphases.

Despite the growing research base and the compelling arguments, however, few contemporary programs of study in school mathematics have heeded such admonitions for change. It is perhaps for this reason that many mathematics education researchers have focused on familiar topic areas (such as those just mentioned; see Davis & Simmt, 2016, for other examples) as means to incorporate studies of complexity into school mathematics. Discussions of and research into possible sites of integration have spanned all grade levels and several content areas, and proponents have tended to advocate for complexity content but in a less calculation-dependent format.

Closing Remarks

For many mathematics educators, complexity thinking might seem like a Pandora's box. If the field were to open it and take up the topic seriously, an array to world-changing possibilities would impose themselves. Complexity thinking challenges many of the deeply engrained, commonsensical assumptions on how humans think and learn. It interrupts much of the orthodoxy on group process and collective knowledge. And, in particular, as a curriculum topic, there is no straightforward way to fit complex modeling into the mold of contemporary school mathematics. It transcends procedures with its invitation to experiment; it demands precision, but in the service of playful possibility; it is rooted in computation but off-loads most of that work onto digital technologies; it requires facility with symbol manipulation, but that manipulation is more for description than deriving solutions. In other words, merely considering complex modeling as a possible topic for today's classrooms forces a rethinking of not just *what* is being taught, but *why* some topics maintain such prominence and *how* topics might be formatted to engage learners meaningfully and effectively.

Indeed, as the example of exponentiation might be used to illustrate, if complexity were to be seriously considered as a curriculum topic, it would compel reexamination of the very foundations of school mathematics. Not only must the "basics" be available for interrogation and revision, emphases of computation-heavy and symbol-based processes would have to be complemented with modeling-rich and spatial-based possibilities. Importantly, this is not an either-or situation. Taking up modeling as a focus of school mathematics does not negate computation and symbolic manipulation, but such a shift does reposition them as means rather than ends.

It will be interesting to see if and when the culture of school mathematics is able to move in the direction of complexity thinking. The discourse itself suggests that, while a sudden and dramatic shift could happen at any time, it is more likely that the grander system will find ways to maintain its current emphases for some time longer. Caught in a tangle of popular expectation, deep-rooted practice, entrenched curricula, uninterrogated beliefs, and lucrative publishing and testing industries, school mathematics is an exemplar of a complex unity. This insight, more than any other, is the one that sustains my interest. Sooner or later, a well-situated wing flapping will trigger that moment of exponential change through a cascade of transformations that pull school mathematics into a new era.

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