Mathematical Thinking and Learning, 14: 55–80, 2012

ISSN: 1098-6065 print / 1532-7833 online
DOI: 10.1080/10986065.2012.625076

Routledge
Taylor & Francis Group

Try to See It My Way[1]: The Discursive Function of Idiosyncratic Mathematical Metaphor

Dor Abrahamson, José F. Gutiérrez, and Anna K. Baddorf
Embodied Design Research Laboratory, University of California, Berkeley

What are the nature, forms, and roles of metaphors in mathematics instruction? We present and closely analyze three examples of idiosyncratic metaphors produced during one-to-one tutorial clinical interviews with 11-year-old participants as they attempted to use unfamiliar artifacts and procedures to reason about realistic probability problems. Our interpretations of these episodes suggest that metaphor is both spurred by and transformative of joint engagement in situated activities: metaphor serves individuals as semiotic means of objectifying and communicating their own evolving understanding of disciplinary representations and procedures, and its multimodal instantiation immediately modifies interlocutors' attention to and interaction with the artifacts. Instructors steer this process toward normative mathematical views by initiating, modifying, or elaborating metaphorical constructions. We speculate on situation parameters affecting students' utilization of idiosyncratic resources as well as how socio-mathematical license for metaphor may contribute to effective instructional discourse.

INTRODUCTION

When students contribute idiosyncratic metaphors to a mathematics classroom discussion, what might be the purpose and utility of these utterances? On the one hand, one might believe, metaphors are "off task" because they often do not include any numerical content. How, as such, could metaphor possibly promote the development of mathematical knowledge? Moreover, encouraging metaphors might preclude effective classroom discussion, given that each student brings to bear subjective experience that is likely to be irrelevant to all other students.

On the other hand, it has been proposed that mathematical reasoning draws on nonquantitative resources (Arcavi, 1994; Case & Okamoto, 1996; Johnson, 1987; Pirie & Kieren, 1994; Presmeg, 1986; Radford, 2006; Resnick, 1992; Thompson, 1993) and that mathematical instruction is effective only to the extent that it engages multimodal imagery (Abrahamson, 2009a; Bartolini Bussi & Boni, 2003; Dewey, 1928/2008; Edwards, Radford, & Arzarello, 2009; Lemke, 1998; Nemirovsky & Borba, 2004; Papert, 1980; Presmeg, 2006). Indeed, dynamical images have been implicated as instrumental to scientific and mathematical creativity, discovery, and invention

[1]Lennon, J., & McCartney, P. ("The Beatles") (1965). *We can work it out*. London: Abbey Road Studios.

Correspondence should be sent to Dor Abrahamson, Graduate School of Education, 4649 Tolman Hall, University of California, Berkeley, Berkeley, CA 94720-1670, USA. E-mail: dor@berkeley.edu

(Benfey, 1958; Black, 1993; Hadamard, 1945; Poincaré 1897/2003; Schoenfeld, 1991; Steiner, 2001; Wilensky, 1997). It thus appears we should not be too hasty in dismissing the possibility that metaphor in fact bears potential utility for pedagogical practice.

To the extent that metaphor may indeed support individual mathematical learning, it would be helpful to understand why and how students engage metaphorical reasoning, and which contexts best enable productive metaphor-based discussion.

Authentic Idiosyncratic Metaphors Are Like Truffles

In this modest empirical article, we report on findings from retrospective analysis of data originally collected during the implementation of a design-based research study. That study, which investigated 9–11-year-old students' probabilistic reasoning, was conducted in the form of individual task-based clinical interviews, in which a researcher probed a participant's problem solving with unfamiliar artifacts (Abrahamson & Cendak, 2006). Preliminary data analysis revealed several instances of metaphor, yet that finding appeared insignificant at the time because only 3 of the 28 individual participants produced a total of just 4 metaphors in over 26 hours of data. Only recently have we returned to these rare occasions to investigate the function of metaphor in mathematics education. Herein, we present and demonstrate an emerging interpretation of the discursive function of metaphor. That said, our scarce data delimit the generalizability of this thesis. Our theoretical rationale is as follows.

Quantitative reasoning evolves in individuals in the form of goal-oriented schematic cognitive structures (Piaget, 1968). Yet this evolution necessarily nurtures from the social context (Roth, 2001), wherein individuals engage cultural artifacts as means of participation in disciplinary practice (Newman, Griffin, & Cole, 1989; Sfard, 2007; Vygotsky, 1934/1962). Teachers are institutionally designated social agents charged with promoting students' development. They do so by shaping students' attention toward and interaction with situations under joint inquiry (Bartolini Bussi & Mariotti, 2008; Goodwin, 1994; Stevens & Hall, 1998). Students, in turn, utilize a variety of personal resources to accomplish curricular tasks. Metaphor, in particular, has been implicated as conducive to meaningful engagement in mathematical tasks (e.g., Lakoff & Núñez 2000; Presmeg, 1986, 1992, 1998). Focusing on metaphor, we propose to highlight its inherently discursive motivation and function. Specifically, we propose that instructors and students utilize metaphor as a discursive means of striving to create or amend shared views, meanings, and uses of artifacts in mathematics learning environments. Mathematics teachers and students, like poets, use metaphor as one means of getting other people to see things their way.

This study may bear implications for policy debates concerning the design and implementation of mathematics-learning materials, activities, and pedagogical supports ("The math wars," Schoenfeld, 2004). Specifically, validation of our thesis concerning the pedagogical utility of metaphor could encourage teachers to cultivate socio-mathematical norms of classroom discourse that embrace and even elicit metaphorical contributions (cf. Ball, 1993; Cobb & Bauersfeld, 1995). More broadly, research on the nature and function of imagery is relevant to debates over the nature of reasoning—whether it is essentially amodal (propositional) or modal (compare Barsalou, 2008; to Dove, 2009; Fodor, 1980; Thagard, 2010; Wilson, 2002).

Research on metaphor in mathematics discourse is related to a larger literature on imagery in mathematics sense-making and reasoning. Whereas it is beyond the scope of this article to survey this vast literature, a brief overview will enable us to hone our thesis.

Imagery and Metaphorical Reasoning in Mathematics Problem Solving

Presmeg (1992) argues for the essential role of visualization in mathematics problem solving. So doing, she weaves together a framework built on her own empirical findings as well as the epistemological philosophy of Immanuel Kant, the cognitive–developmental theories of Jean Piaget and Bärbel Inhelder, and the cognitive linguistics theory of Mark Johnson and George Lakoff. This section presents key constructs from this synergistic perspective so as to position our thesis in the literature.

Individuals' mathematics reasoning engages multimodal dynamical images that are visual, auditory, and kinesthetic perceptions from their experiences. Yet it is not the rich images themselves that serve in mathematics reasoning but rather impoverished traces of these actually experienced images. These traces—cognitive structures called pattern images or image schemas—essentialize the source perceptions into their common distinctive attributes, such as recurring spatial configurations and dynamical relations among their elements. As such, the schemas can organize experientially disparate phenomena systematically into clusters, which inform the organism's contextual functioning by lending potentiality of meaning and response (see also diSessa, 1983, 1993).

In terms of their apparent concreteness, image schemas are theorized as positioned in between situations and concepts. As such, image schemas play bilateral bridging roles between situations and concepts: through the cognitive mediation of image schemas, problem solvers both evoke candidate conceptual knowledge and apply it to novel situations they are trying to make sense of (i.e., to model, generalize, mathematize). Accordingly, Presmeg (1986, 1992) identifies two reciprocal functions of image schemas: concretizing the abstract and essentializing the concrete. For example, image schemas enable a person who is engaged in making sense of a problem situation to consider interpreting the situation as a "case of" by filtering out irrelevant contextual details of the sensorial percept (cf. van den Heuvel-Panhuizen, 2003). Here one might note that image schemas are not static cerebral structures but emergent sense-making procedures (Glenberg, 1997; Roediger & McDermott, 1993). Moreover, these schemas have been implicated as embodied: "[T]hat of which we think emerges from and in . . . [perceptuo-motor] activities themselves" (Nemirovsky, 2003, p. 109; see also Rotman, 2000).

Image schemas vary along several dimensions relevant to the study of mathematics learning. Whereas the schemas may be canonical, such as balance images of algebraic equations, other images, such as viewing a graph's *x*-axis as the horizon, may be idiosyncratic. Also, the evocation of image schemas may serve either "reproductive" (mnemonic) or "productive" (creative) purposes of mathematical reasoning. Finally, Presmeg implicates analogical reasoning as the cognitive means of applying images to particular situations, and metaphor as the linguistic embodiment of analogy. In sum, "The power of metaphor lies in its use in making sense of new conceptions in terms of already existing conceptions—thus in 'building new conceptions'" (Presmeg, 1998, p. 29, see also Fauconnier & Turner, 2002; Sfard, 1994). To support her views, Presmeg (1986, 1992, 1998) presents empirical data of students citing metaphor-based heuristics in solving mathematical problems or interpreting mathematical artifacts.

Implicit to Presmeg's examples of students' idiosyncratic metaphorical images is an apparent assumption that her study participants had previously evoked these very images to reason about the same particular mathematics subject matter, possibly through prior application to contextually similar situations. We thus remain with the question as to how and why mathematics students first metaphorize images. This research question is thematic to mathematics-education research on the "birth of metaphor" (Sfard, 1994) or "sprouting of signs" (Radford, 2003), which are significant moments in the ontogenesis of mathematical concepts (Abrahamson, 2004).

Taking on the question of how metaphors are first articulated, we will be inquiring into a set of empirical episodes to evaluate the following conjecture.

> When mathematics students attempt to make sense of new phenomena—situations they are expected to interpret and proposed artifacts for doing so—they try to determine what these phenomena are like, what they are analogous to. In this sense-making process, students recall familiar phenomena. To communicate their insight—how they are seeing the phenomena—students use metaphor.

As our data examples will demonstrate, mathematical reasoning may demand creative analogical construction. In cases of joint inquiry, metaphor serves as a "frame" (Fillmore, 1976) for orienting the interlocutor's view toward the problem situation (Stevens & Hall, 1998). Metaphor can thus be understood as an internally available means of objectifying emergent understanding. We thus also wish to complement previous suggestions by which mathematical learning is the semiotic appropriation of externally available artifacts (cf. Abrahamson, 2009b; Radford, 2003; Sfard, 2002).

METHODOLOGY

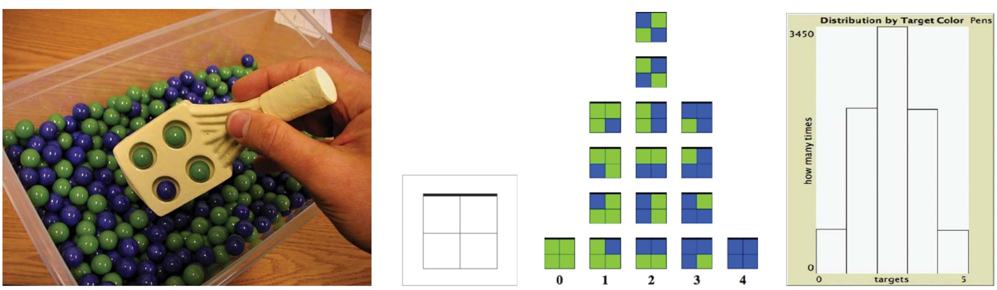
The original project whose data we are investigating in this post hoc study was conducted in the design-based research (DBR) approach (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). Reports on DBR studies often do not achieve satisfactory coherence among theoretical perspectives, instructional design, and a posteriori analysis (Abrahamson & Wilensky, 2007; Artigue, Cerulli, Haspekian, & Maracci, 2009; Kelly, 2004; Puntambekar & Sandoval, 2009; Ruthven, Laborde, Leach, & Tiberghien, 2009). We attempt to achieve such coherence as follows: (a) our post-hoc analytic attention to the multimodality of discourse in our empirical data resonates with the epistemological commitments underlying our embodied-design framework (Abrahamson, 2009d); and (b) probability is a suitable content domain for the study of multimodal reasoning, because the challenges of expressing randomness numerically may impel students to resort to alternative discursive genres such as metaphor (Rubin & Hammerman, 2007).

An appeal of the DBR approach is that students' metaphorical reasoning was captured as they were possibly first constructing a new idea (in contrast with experimental design explicitly engineered to elicit student metaphor, e.g., through a survey, see Groth & Bergner, 2005; cf. Rubin & Hammerman, 2007). A tradeoff of our study's ecological authenticity, however, is the scarcity of relevant data. We partially address the consequent lack of statistical power with in-depth qualitative analyses.

The data excerpts cited next are from interviews with three 11-year-old students from a private suburban school. The context of these interviews was a DBR study of late-elementary and middle-school students' intuitive probabilistic reasoning. We also evaluated the potential roles that various technology of our design could play in enabling students to coordinate intuitive and mathematical formulations of the focal situations (Abrahamson & Cendak, 2006). Using a flexible protocol, the first author and a research assistant engaged each participant in a task-based semistructured clinical interview (diSessa, 2007; Ginsburg, 1997; Goldin, 2000). The sessions were videotaped for subsequent analysis, and selected episodes were transcribed. The research team employed collaborative microgenetic analysis techniques (Schoenfeld, Smith, & Arcavi, 1991; Siegler & Crowley, 1991) as well as the grounded-theory approach (Glaser & Strauss, 1967), through which behavioral patterns were identified (diSessa & Cobb, 2004) and iteratively crosschecked and developed vis-à-vis the entire data corpus.

A methodological challenge we encountered in analyzing the video data was the liability that the interviewer–researcher (the first author) revise history by introspectively ascribing to himself thoughts he had not in fact entertained during the interview several years prior. Our field notes and minutes from our daily electronic communications during the data collection proved helpful in addressing this challenge. Wherever doubt still remained, we either omitted the interpretation or qualified it.

Figure 1 depicts the design's focal artifacts. The marbles scooper is used to draw exactly four marbles from an urn of mixed green and blue marbles (see Figure 1a).[2] We ask the study participant to guess, "What will happen when we scoop?" (see Abrahamson, 2009c, for dimensions of deliberate ambiguity in this prompt). Next, the participant is guided to use crayons as well



a. *b.* *c.* *d.*

FIGURE 1 Materials used in the study—experimental and analytic embodiments of the 2-by-2 mathematical object: (a) the marbles scooper; (b) a template for performing combinatorial analysis; (c) the combinations tower—a distributed sample space of the marbles-scooping experiment; and (d) an actual experimental outcome distribution produced by a computer-based simulation of this probability experiment. (Color figures available online.)

[2]Viewed in "slow motion," the experiment is revealed to be not binomial but hypergeometric. That is, when the first marble is scooped, the urn has one less of that marble's color to be scooped into the remaining three concavities, so that the experiment is strictly a case of "without replacement." Nevertheless, the experiment can be viewed as functionally approximating the binomial due to the minute ratio of the sample size (four marbles) to the content of the urn (hundreds of marbles).

as a pile of stock-paper cards each bearing a blank 2-by-2 matrix (Figure 1b) so as to create the sample space of the experiment and assemble it in the form of the combinations tower (see Figure 1c). Later, the interview moves to computer-based simulations of the same experiment (e.g., see Figure 1d).

Elsewhere, we furnish detailed explanation of the design motivation and rationale and report on findings and design modifications throughout the iterated study cycles (Abrahamson, 2009a, 2009b, 2009d; Abrahamson & Cendak, 2006; Abrahamson & Wilensky, 2007; Abrahamson, Janusz, & Wilensky, 2006). In the current retrospective study, we seek to focus on instances of metaphorical reasoning so as to understand their emergence and evaluate their function and effect within instructional situations. So doing, we are evaluating a conjecture regarding the pedagogical role of metaphor: participants in mathematics instruction—both teachers and students—generate metaphors as discursive means of concretizing their perceptions of activity elements, including problem situations, disciplinary and pedagogical artifacts, and solutions procedures.

RESULTS AND DISCUSSION

We are evaluating the conjecture that spontaneous metaphor plays an important function in fostering mathematics learning. Specifically, the discursive form of metaphor enables participants in mathematical conversations to shape mutual views of phenomena, such as situations, objects, and artifacts, by going beyond naming what they are seeing to communicating how they are seeing it. So doing, we maintain, participants promote their own learning: (a) directly, through articulating and refining mathematical assertions; and, (b) indirectly, by creating opportunity for formative assessment that is uniquely attuned to their idiosyncratically evolving schemas for the mathematical subject matter.

In this section, we present and discuss empirical data exemplifying spontaneous metaphor in conversations around task-based instructional activities that were designed to promote mathematics learning. The three episodes, each from an interview with a different student, span a range of conceptual issues targeted along the interview protocol. Perhaps coincidentally, the episodes thus became ordered also with respect to the apparent elaborateness of the metaphor. In all three episodes, we argue, members of the dyad used metaphor to get the other person to see things their way, where the "things" will be ambiguous mathematical artifacts. Following the presentation of all three episodes, we offer summative comparison across the episodes. We then conclude the article with evaluative remarks concerning the function of spontaneous idiosyncratic metaphor in mathematics education. (Video clips of all three episodes can be viewed here: http://edrl.berkeley.edu/publications/journals/MTL/Abrahamson-MTL/.)

Serpentine Combinatorial Analysis: Challenges of Generalizing Contextual Metaphor

We join Jake (pseudonym), an 11-year-old and reportedly "low-achieving" male student, and the researcher at a point along the interview protocol where Jake is attempting to color in the cards

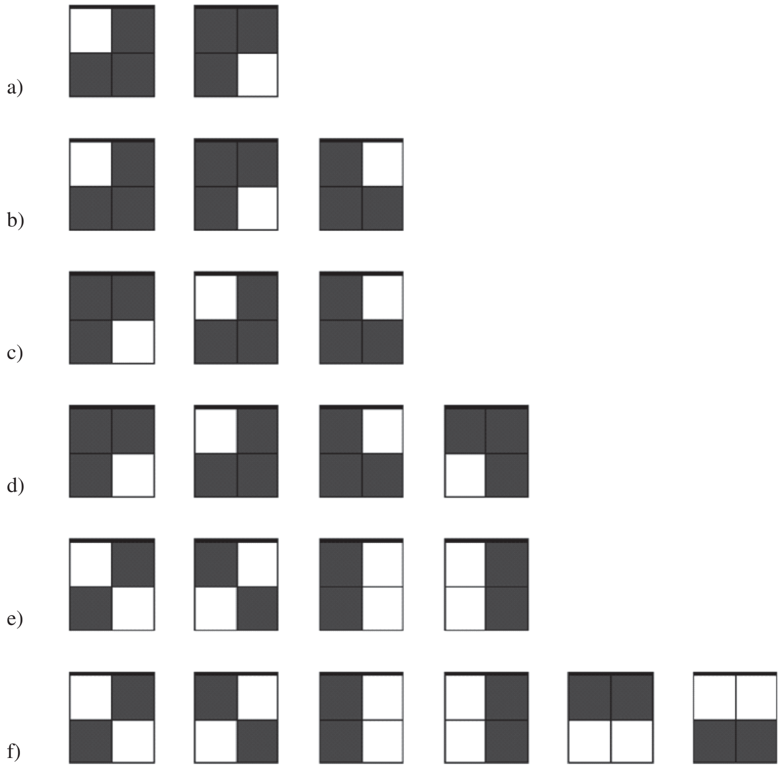


FIGURE 2 Selected phases in Jake's guided combinatorial analysis of the three-blue combination (Figures 2a–2d) and the two-blue combination (Figure 2e, 2f). The darker line on top of each card was designed as a means of supporting the analysis by making rotations distinct.

so as to show "what we can get when we scoop." Prior to this episode, Jake had created four of the six possible permutations on the two-green-and-two-blue combination (hence "2g2b"). Currently, Jake is working on the one-green-and-three-blue combination (hence "1g3b") and has so far created two of four permutations (see Figure 2, where we use white-and-grey instead of green-and-blue, respectively, for print clarity).[3]

Jake: <00:24:44> I think, mayb– . . . [intently gazing at the two cards] /4 sec/ Well, they're like L-shaped, and I don't think you can get any more L-sh. . . . Actually! . . . Yeah, you can't get anymore L-shapes. Oh wait, yeah you *can*, you *can* [lifts crayons, completes a third card; now three cards lie in a row before him (see Figure 2b)] /11 sec/ . . . . Uhm. . . . /29 sec/ [gazes at the three cards, rearranges them on the

[3]For discussions of research on students' challenges and possible trajectories in learning to perform combinational analysis, see Batanero, Navarro–Pelayo, and Godino (1997), English (2005), and Sriraman and English (2004).

desk (see Figure 2c); with his right hand (RH) index finger, Jake gestures curves above each card] . . . That's it.

Dor: Ok, and how would you prove that that's it? Is there a way? [. . .]

Jake: Well, they're all L-shaped, and well I kinda like made, kinda like a pattern like, it's like a snake, it's going "dun dun dun" [each "dun" is accompanied by pointing at one of the three darker cells in the L figure, in the left-most card], and then it's going swivel around, and like that [RH index glides over the cards in a curved meandering motion, highlighting the L figures in each of the three cards; the motion appears to follow a continuous line, such that the end of each L roughly connects or skips to the beginning of the L in the next card].

Jake used an L-shaped configuration as a material anchor for a conceptual blend (Hutchins, 2005)—he blended a snake schema into a specific L-shaped concatenation of three contiguous spatial elements. Moreover, Jake "elaborates the blend" by importing into the semantically impoverished L-shape dynamical behavioral properties typical of the more semantically rich snakes (see Gentner, 2001). He introduces the snake as a simile ("it's like a snake") and then, implicitly, as a metaphor ("it's going . . ."). The snake blend animates the static L-shape, mobilizing it to move snake-like. So doing, Jake interprets the several spatially contiguous cards lying on the desk as though they inscribed a picture story advancing from left to right. That is, Jake sees the three frames as each depicting the snake's location in temporally consecutive instances. And yet, whereas Jake ran his blend in an aesthetically appealing pattern, this run was insufficiently systematic as a combinatorial-analysis strategy, because Jake could not determine whether he had exhausted all the possible snake positions in the card template. In particular, whereas Jake's snake swivels, it is not clear how careful Jake was in monitoring the snake's progress along each and every discrete rotation (see in Figure 2c how the first shift, from the left-most card to the middle card, skips a rotation).[4]

The interviewer (Dor), who sees that Jake has created only three of the four 1g3b permutations, nevertheless recognizes semiotic potential in the snakeness Jake has just blended into the cards. Specifically, Dor recognizes that the blend could support Jake in coloring in the cards in a way that would support the didactical objective of students experiencing and appreciating the utility of a systematic approach to combinatorial analysis. Dor's diagnosis was that Jake was not equipped to notice inconsistency in the snake's progress from one frame to the next. Dor thus wishes to introduce rigor into Jake's own blend in a way that would enable Jake to find the fourth permutation (the missing rotation)—Jake would thus succeed both locally in completing the analysis of the 1g3b combination and, more generally, in experiencing the systematicity of combinatorial analysis. In order to introduce rigor into Jake's strategy, Dor is about to highlight for Jake a unique figural element within each L-shape—the middle of the three squares; Dor will code this square as the "corner" (see Goodwin, 1994, on "highlighting" and "coding" in experts' mediation of professional vision; see Mariotti, 2009, on "focalizing"). Dor hopes that Jake will

[4]Jake's modest discovery faintly echoes Friedrich August Kekulé von Stradonitz's momentous metaphor-based discovery of benzene's ring-shaped structure. Kekulé apparently dreamed of a snake swallowing its own tail (Benfey, 1958). In Abrahamson, Wilensky, and Janusz (2006) we discuss a case of middle-school students inventing similar figural strategies to analyze a 3-by-3 green/blue table. Although they discovered many of the 512 possibilities, the students ultimately abandoned the strategy, because they had no clear criteria for determining whether they had exhausted the sample space.

thus be better able to notice that the swiveling L had skipped a rotation. As we shall see, however, manipulating perception can be difficult, precisely because students do not know the didactical objective of an instructional action.

Dor: So where is the corner of the snake?
Jake: Corner of the snake? What do you mean corner of the snake?
Dor: [LH thumb and index make an L icon] It's like a, it's like an L.
Jake: [simultaneously] Yeah, it's L-shaped.
Dor: Where is the corner of the L? [Jake points to the corner of the L on the first card and then looks at Dor] There.
Jake: Yeah.
Dor: Where's the. . . . Where is it next? [Jake indexes the L's corner on the next card] There [Jake indexes the corner on the third card] and here. Now, as it moves [points with pen]—as it twists [pen curves over the L]—look, the corner is here. . . .
Jake: Oh, wait. There should be four. [reaches for another card and colors it in] /20 sec/ [completes coloring in the card (see Figure 2d) and puts the crayon down] Ok. /2 sec/ So, now I *know* there's no more, uhm, 1-greens cause the corner's in all places [touches the corners in each of the four cards].

Thus, Jake and Dor co-created a locally effective combinatorial-analysis strategy—Jake saw a swiveling snake, and Dor tamed it so that it visits all rotations. Particularly aligned with the didactical goal is Jake's assertion that he now knows he has exhausted the local analysis. However, might this metaphor become a useful and generalized reasoning tool? That is, Jake may well regard the snake metaphor as a local strategy to solve just one part (1g3b) of the problem, a strategy that does not necessarily generalize to other parts. Can Jake apply this improved blend to a different context, such as the 2g2b set? Although a two-squared snake in the 2g2b set is not L-shaped, it might still be envisaged as swiveling around the card and thus support Jake's combinatorial analysis of 2g2b. Will Jake transfer the swivel strategy from 3-celled figures to 2-celled perceptual variants? To begin with, could Jake see a 2-celled concatenation as a snake at all, or will the snake analogy prove too "brittle" (Clement, Brown, & Zietsman, 1989)? If so, how might Dor intervene so as to support Jake in re-evoking the serpentine combinatorial analysis?

Several minutes later, Jake is attending to the 2g2b set. With Dor's organizational support, Jake has created four of the six possible cards, including the two X-shaped patterns (see Figure 2e). As we shall see, Jake appears to believe there are only four cards in the 2g2b set, possibly because there were only four cards in each of the previous two sets he had created (1g3b and 3g1b, see below). Accordingly, when Dor prompted Jake to search for more permutations, Jake attempted to rationalize his work as complete. Jake began by referring to the cards' symmetry as supporting his position.

Jake: <00:28:50> I kind of [inaudible], like, kind of like diagonal-wise, and they're on both sides [each hand gestures to one of the X-shaped patterns].
Dor: Ok, so you feel you've *done* the diagonals.//
Jake: //Yeah, and also, I did, I did it on each half [each hand gestures to one of the two other cards (see Figure 2e)]
Dor: Ok.

Jake: Yeah, so . . . /2 sec/ um, so I don't really think there's anything left. Also, there's four spaces [four cells in an un-colored card template], and there's four of these [gesturing to the four cards].

At this point, Jake believed he had exhausted the 2g2b group. His final assertion was that there are four cells in the card, and that he had discovered four permutations. Enter snake.

Dor: Ok, so, so, let me, uhm [brief chuckle] remind you of th–, of your own kind of thinking. So there seems like, there, you have two tricks here. One is doing it like . . . uh [RH makes a flip-flop "symmetry" gesture], the uh, sort of [Jake keeps interpolating "yeah"] . . . yeah ok, but once you find a certain, a certain thingy, a certain configuration or whatever, then you do the mirror image or flip image or whatever you want. . . . So you did the same here—these are kind of flip images of each other [rearranges set of four 2g2b cards to twin each card with its mirror image (Figure 2e depicts the eventual order)]. But then there was another thing you did *here* [indicates the stack of 1g3b cards treated earlier], which you haven't done here [indicates the 2g2b cards], and that was the idea of the going [RH gyrates clockwise at wrist, grip fixed, "opening a jar"] . . . of the snake going around.

Jake: Oh, yeah! [sits up]//

Dor: // . . .which is kind of different.

Jake: Uhm . . ./15 sec/ [rearranges the cards on the desk several times] well, but the corner. . . . No. Yeah. No. There's two of them. /4 sec/ I can't do the snake thing, cause there's no corner, cause there's two of them.

Dor: Ok, so how about a *short* snake [laughs].

Jake: Um . . . /13 sec/ I don't know if the corner would be on this side or this side, though [gesturing toward two adjacent blue cells, indicates one, then the other].

Dor: Ok, so let's just think of this idea of, of rotating around the center [RH performs the jar-opening gesture] cause that's essentially what you were doing here [indicates the 1g3b stack]. You were saying that the L is sort of rotating [bends RH index to make an L-shape, then rotates the L] around this middle point [of the card; RH index beats card center four times].

Jake: Yeah.//

Dor: //And we know we found . . .

Jake: //Oh!!! . . . [abruptly sits up] Oh! So the green's kind of like switching around [focusing on a 2g2b card with adjacent green cells, RH index and middle finger form a fixed co-pointing pair that then rotates 90 degrees around the center, thus representing the rotation that had been missing between the two symmetrical permutations, see Figure 2e], but these are just kinda different, so the green would be . . . ha! . . . [takes a card from the stack and colors it in].

Dor: Bingo . . . /8 sec/ [Jake takes another card from the stack and fills it in (see Figure 2f)] Ok.

Jake: Now I think I have them.

Dor: Yeah.

Jake: Ok!

Jake's difficulty in transferring a strategy he had learned for the 1g3b set over to the 2g2b set, it appears, was due to overprominence of the "corner" perceptual feature in this strategy—a 2-green snake has no corner, and so there was no figural cue in the 2-green concatenation to alert that it, too, affords swiveling from one permutation to the next, as the 1g3b L-shaped snake had done. Only once the focus shifted from the snake's "corner" to the card's "middle point," it appears, could Jake see the two squares anew ("a short snake") as affording a variant on swiveling ("switching"). Had Dor initially established the snake's "head" rather than its "corner" as the figural cue, perhaps Jake would have subsequently transferred the snake strategy without Dor's considerable support. Moreover, snakes appear to move forward from their head, not their center, such that preferring the head over the center as the figural cue would have been more compatible with Jake's metaphorical blend. However, the "corner" appeared to Dor a useful cue at the time, because it preempted any ambiguity as to whether a given end-cell were the snake's "head" or "tail" (to appreciate this tracking challenge, return to Figure 2c).

In sum, Jake's episode described a "metaphor breakdown" that a student experienced during a guided attempt to transfer a perceptual heuristic he himself had devised. Namely, the transfer context did not readily afford for the student an application of his own metaphor, and this application may have been further hampered by the tutor's heuristical elaboration of the student's metaphor when it was first developed. Ultimately, however, the metaphor-based discourse appears to have been pedagogically useful because it enabled the tutor to steer the student toward the didactical objective (notions of systematicity and rigor that are requisite features of combinatorial analysis).

The utility of metaphorizing an idiosyncratic image schema in a local context thus does not guarantee its transferability to conceptually proximate contexts (see also Clement, Brown, & Zietsman, 1989). Yet students cannot anticipate the prospective utility and generalizability of a locally effective informal metaphor of their own design, simply because they are not yet privy to the conceptual field in which they are operating. The teacher, who attempts to guide students' re-invention of powerful mathematical schemes, can intervene by "tuning" students' own image schemas in accord with the didactical goals. This "tuning" is accomplished through discursive actions referring selectively to the images' structural dimensions. From the vantage point of disciplinary expertise (or, at least, from the vantage point of pedagogical content knowledge), the teacher deemphasizes the image's local properties that hinder contextual generalization and emphasizes its didactically critical, transferable properties.

I'm Just a Machine: An Epistemic Metaphor Isolates Chance and Cognition

We now turn to the case of Sima, an 11-year-old high-achieving female student. She and the interviewer (Dor) are at a point along the interview protocol where they have completed the construction of the combinations tower (see Figure 1c) and are currently examining its relations to the marbles-scooping experiment.

Sima and Dor have been discussing the meaning of the sample space with respect to experimental outcome frequencies. So doing, they have been making headway in naming and distinguishing two different views of the sample space: either as 16 equiprobable elemental-event specimens or as 5 heteroprobable aggregate-event sets. And yet, it appears, the interlocutors are encountering difficulty in agreeing on the referents of their utterances. More precisely, whereas they agree on which objective things they are referring to, they tacitly differ on how they are seeing these objects and, therefore, on what these objects are. We now elaborate.

Dor and Sima bear different default ways of seeing each of the individual cards; for example, the 1g3b card with green in the top-right cell. When Dor looks at this card, he sees by default 1 of 16 elemental events with a unique spatial configuration and a 1/16 chance of random selection, whereas Sima by default does not attend to the card's particular green–blue configuration and so she sees an aggregate event with a 4/16 chance. Sima is not necessarily making a conscious decision to ignore the order of green and blue—she is well able to switch to Dor's view, but whenever she is not prompted to see it his way, she sees it her way. When the tacit disagreement is irrelevant to the conversation, the conversation flows, but whenever the meaning of the card is pertinent, the conversation goes awry or even breaks down.[5]

During the interview itself, however—several years before this analysis was proposed—Dor interpreted the confusion not as a deep cognitive phenomenon but as a superficial semantic issue to be settled before any substantive mathematical conversation could take place. Improvising, Dor chose for each of the 16 cards human names, such as "Joe" and "Carol," and assigned them to five families. Dor's intention was to shift the discourse into an amathematical domain, as a means of drawing on Sima's informal understanding that objects can be viewed alternatively either as unique or as members of a category. Moreover, Dor initiated a new activity: he flipped all of the cards over, so that their blank side showed, and suggested marking the cards with the 16 letters A through P. Dor further evoked a hypothetical scenario involving a third person who happened to walk in on the interview; finding all 16 cards labeled A through P, this person would not be told that the cards bore color patterns on the concealed side. What might this person think about the relative chances of randomly drawing either the K or D card out of these 16 cards?

Sima responded that as "letters", the 16 cards would have equal chances of being selected, but that if they were flipped, these very same cards would have different chances of being selected, in accord with their color pattern "families," and irrespective of their "first names." But, then Sima said that even as A-through-P alphabetical characters, the cards have different chances, for example K-cum-Joe, a 2g2b family member, would have a better chance than D-cum-Carol, a 3g1b.

Dor: <00:49:32> So I have 16 cards, and I mix them up randomly, and you think that "K" has a better chance of coming up than "D?"
Sima: Ahuh [affirmative nod]
Dor: Huhhh /5 sec/
Sima: But when you're talking about K and D—no, because K and D are both just 1-out-of-16. But when you're talking about Joe and Carol—and not K and D—they are "2 green, 2 blue," "3-green, 1 blue."

Sima has expressed what appears to be a paradox—that one's chances of randomly selecting a particular card depend on whether one knows to which subgroup in the combinatorial space the card belongs.

[5]It is as though Sima and Dor are both staring at Jastrow's proverbial duck–rabbit ambiguous image and arguing not over what it is but whether this thing should be fed carrots or fish (Abrahamson et al., 2009; Rowland, 1999). See also Newman, Griffin, and Cole (1989) on the enabling "looseness" in a novice and expert's joint attention to objects used for the enactment of disciplinary cultural practice.

Perplexed, Dor seeks a means of disengaging Sima from the aggregate view. He evokes a brainless random device—a mechanical grabber-hand that slides at a height then descends toward the card pile and picks out any one card:

Dor: But what if I'm just a machine. . . . I don't know what they're called. . . . I'm just a machine—my job is to come and, "Ehhhhh." [eyes dimmed, mimes and ventriloquizes a device blindly selecting from the cards]
Sima: Then it would just be . . . then K and D don't have. . . . Then K and D are just two . . . one . . . they're just . . . they're equal. Equivalent.
Dor: [flipping over all sixteen cards to reveal colors] And if. . . . And if they're this side . . . this way, and I'm still. . . . I'm just a machine, and I come, "Ehhhhh" . . . just picking one randomly . . . . Does . . . Does Joe or Carol have a better chance? /2 sec/ I'm just a machine . . . .
Sima: I guess not.
Dor: Huh!
Sima: /2 sec/ When you're just talking about *independent* little cards.
Dor: *Independent* little cards.
Sima: Yeah, and they have their own I.D.
Dor: Their own I.D.
Sima: Not talking about families—yeah!
Dor: Huh!
Sima: There is no difference.
Dor: Ok.
Sima: They're equivalent.
Dor: There seems like there's this tension, where we're working back and forth between these things. As individual "people cards" you say they have the same chance . . . .
Sima: [nods affirmatively] Um'hhmm . . . .
Dor: . . . but as members of families . . . so some families have a better chance.
Sima: Yeah [nods affirmatively].
Dor: Can you live with both things? Because we'll have to move back and forth between them.
Sima: Yeah [nods affirmatively].

In sum, this episode described Sima and Dor's struggle to achieve consensus over the meaning of the cards on the desk. The interpretive disagreement between the interviewer and student, it thus appears, was not only semantic but touched on the deep conceptual issues: Sima does not see order by default, because she does not see different permutations as actual discernible things in the environment (e.g., Bamberger, 1999).

Moreover, the cards were designed as a pedagogical support for students to understand the experiment's sample space, yet surreptitiously the set of 16 cards became a secondary sampling space—an open urn of sorts lying flat on the desk and containing 16 objects from which to draw (a mathematical commensurability for a $p = .5$ approximated binomial). Managing the card collective's shifting role as either sample space or sampling space can be confusing, moreover coordinating this shifting view with an interlocutor. One is consequently liable, as Sima, to state that two cards, a 2g2b and a 3g1b, compare alternatively as equiprobable or heteroprobable: they are equiprobable as two independent material objects placed in a sampling space but heteroprobable as two naïve views of the marbles experiment.

The machine metaphor appears to have addressed a tacit interpretive conflation bearing significant conceptual repercussions. Namely, by stating, "I'm just a machine," the interviewer compartmentalized two aspects of his agency in the random experiment: (1) the physical action of randomly selecting one of the cards viewed as elemental events; and (2) the mental action of mathematically inferring the expected frequency of the cards viewed as aggregate events. The blind machine—Dor's dummy—could not see cards as aggregate events, so that in the machine context the cards were temporarily denuded of their primary-experiment signification and rendered meaningless objects of equivalent neutrality. As such, this episode has demonstrated a spontaneous idiosyncratic epistemic metaphor that analogized the random process itself. Sima, by temporarily adopting the fictitious machine's point of view, came to see things as Dor saw them.

A Storm in a Teacup is But a Drop in the Ocean: An Idiosyncratic Metaphor Serves as a Discursive "Lemma" for Building an Elaborate View of a Mathematical Artifact

Our final case involves Li, an 11-year-old high-achieving male student. Li and Dor are at a point along the interview protocol where they have already constructed the combinations tower and have discussed its relations to the marbles-scooping experiment (see Abrahamson, 2009b, for analysis of the earlier part of the Li interview). Now Li and Dor have begun working on a computer-based experiment that generates histograms such as in Figure 1d (to interact with an applet of this simulation, visit here: http://ccl.northwestern.edu/netlogo/models/run.cgi?4Blocks). Dor has just introduced the computer model as constituting a simulation of the marbles experiment by highlighting general correspondences between the real and simulated settings.

Idiosyncratic metaphors as personally available semiotic means of objectification

In explaining the simulation to Li, Dor took several virtual samples one by one ("Go Once"), each time pointing out a change in the histogram that recorded the outcome. Dor's explanation focused on the five columns in the virtual histogram and in particular on how they "grow" vertically. Satisfied that Li had understood the analogy between the physical and simulated experiments, Dor is now about to guide Li to monitor an experiment that rapidly selects thousands of random samples one-by-one. (The bold font is meant to enable the reader to selectively attend only to verbal utterance).

Dor: <00:51:30>**So what do you think might be the shape of the columns, as it goes up?**
Li: [Gesturing to the combinations tower on the desk] **Something like this**.
Dor: **Well, let's see.**

Dor activates the experiment. As it is running, he explains how the histogram on the computer screen keeps adjusting its maximal *y*-axis value so as to accommodate the columns' vertical growth. Namely, when the columns extend up all the way to the top of the histogram window, the ordinate recalibrates the *y*-axis scale values so as to accommodate this growth. What one sees is that the tallest column remains "glued" to the top frame, while the other columns adjust accordingly. These adjustments often entail some of the other columns sagging down, as though they have "lost" samples. This "autoplot" function is an elegant technical solution used in many simulations built in the NetLogo computer-based modeling-and-simulation environment (Wilensky, 1999), and yet the elegance appears to trade off with some initial familiarization.

Dor: **So what's happening?**

Li: **It's** [the distribution] **hovering at around this** [gestures again to the tower] . . . . [20 seconds later, when 5,000 samples have been drawn] **Look, it's** [the histogram] **almost exactly like this** [the combinations tower] . . . . [30 seconds later] **Now they're** [columns] **moving less** [inaudible].

Dor: **Why is it moving less?**

Li: /2 sec/ **Because** . . . /3 sec/ [gazes up to the wall] **the larger number** . . . /3 sec/ uhh . . . ["checks in" with the interviewer] /2 sec/ [rapidly] **Like if you have a little glass** [iconic gesture: LH cups a glass in natural position; gazes at gestured glass] **of water and you drop a marble in** [iconic: LH uncups, rises, mimes dropping a marble], **it's gonna be** . . . **there's gonna be, like, a splash** [LH, palm up, abrupt vertical rise], **but if you have a big giant lake** [LH, palm up, drawn above and behind head to encircle giant lake; RH, in jacket pocket, budges as if to complete circumference], **and you throw a marble in** [LH catapults marble, then scratches nose]**, there's just gonna be a ripple** [joins LH thumb and index; lowers hand to chest height on right-side of embodied space; taps fingers together to mark the marble's contact with surface, then inscribes smooth horizontal line across to the left; hand opens]. **It's a, it** . . . [deictic gesture toward the histogram, orients gaze; pivots body towards histogram]

Dor: **Oh, ok. Like each individual additional sample** [LH gyrates swiftly at wrist to indicate iterated addenda (cf. Abrahamson, 2004, p. 794)] **is causing less of a** . . . [LH & RH "contain" the outline of a combinations tower]

Li: [simultaneously] **Yeah, it's like a batting average in baseball.** [6] **If you've already had 500 at-bats** [LH opens, shifts slightly to the left, palm up, "holding" the 500 at-bats, then relaxes], **and then you get out one more time, it's not going to make it go down that much—it'll make it go down like two points** [LH pinches thumb and index as if to reduce quantity, corresponding to the subtraction of two points (cf. Badets & Pesenti, 2010)—it is the same "pinch" gesture he produced for "ripple"]**, or** . . . .

Dor: **Ok, so as we go along, each successive sample causes less of a commotion**.

Li: **Yeah**. [Gazes back at the computer screen] **Look, they're just barely moving.**

[6] A "batting average" is the ratio of hits (successful batting) to at-bats (opportunities to do so) over a given period of time. This "average" is usually expressed as a decimal quotient, and it can be updated by adding appropriately to the numerator and denominator. At the beginning of a season, the "average" changes erratically, but as the season progresses, it stabilizes.

From a mathematical point of view, the conversation pertained to the diminishing proportional effect of single samples on the distribution of actual experimental outcomes. Viewed on the computer screen, the compressed histogram stabilizes toward the expected 1–4–6–4–1 distribution, dynamically embodying the law of large numbers. Eventually, the recorded changes within each column category are smaller than the value represented by a single pixel on the screen, so that the structure no longer jitters and appears static (albeit the ordinate's maximum value is constantly updating, thus revealing that the process remains active).

The transcription supports Dor's in situ impression that Li intuitively understood the principle of diminishing proportional impact. We further see that Li's limited fluency with rational numbers and their common linguistic forms did not enable him to communicate his intuition using normative vocabulary, such as "proportion" and its cognates, as witnessed in his aborted attempt, "Because the larger number . . . ." In response, Li evoked situations that enabled him to express his view of the simulation. Expanding on Radford (2003), the metaphors constituted for Li personally available idiosyncratic semiotic means of objectifying a multimodal image schema.

Metaphor as a discursive lemma for recalibrating interpretive schemes

The previous transcription includes two situations allegedly analogous to the histogram: a jettisoned marble disrupts a smooth water surface and an at-bat changes a batting average. In both analogies, some voluntary action results in perturbation to a (material or numerical) aggregation. Moreover, within each analogy is a differential impact this action has on small versus large aggregates (albeit the baseball analogy elides explicit reference to the small quantity structure; see Schegloff, 1996, on the abbreviated inferential form of enthymeme). The logical structure of the argument within each situated context is of the general form $a/b > a/c$ because $b < c$. Namely, the overall impact is greater in the case of a marble in a glass (splash) as compared to the case of a marble in a lake (ripple); and the impact of a single "out" at-bat on the batting average is smaller later in the season.

We wish, at this point, to develop the explorative conjecture that Li's water analogy—and moreover, his temporary embodied shift away from the artifact to a discursive space wherein he built this analogy—was spurred by breakdown in his flow of reading the histogram. Li's tacit expectation had been that the outcome aggregate would continue growing up in constant spatial quota—that for every successive sample there would be a constant diagrammatic change—just as he had witnessed in the first several explanatory samples. However, his actual sensory perception was of a diminishing and eventually vanishing diagrammatic change. When suddenly faced with this anomalous behavior, Li lost his "grip on the world" (Merleau-Ponty, 1958/2005)—his schemes for seeing the experiment through this technologically sophisticated mathematical artifact lost traction. The analogies he subsequently authored, we maintain, constituted Li's (successful) attempt to amend this felt discrepancy between the expected and the evidenced—the analogies served as means of accommodating his schemes so as to assimilate the autoplotting histogram.

Li's technique for recalibrating his view of the histogram drew on tacit perceptual fluency with perspectival aspects of imagery. Namely, how might one reconcile the knowledge that a quantity is growing with the sensory perception that its retinal image is by-and-large constant? This reconciliation can be achieved by projecting the quantity further away in the visual field. If the histogram is perceived as receding further away, then we tacitly expect each successive accretion of a constant quota to result in a smaller and smaller imprint on the retina. Perceived as a larger space, say a larger body of water, the compressed histogram once again makes sense to Li. More precisely, Li once again makes sense of the histogram.

Thus, Li's imagistic constructions of the glass and the lake are adjusted to natural "visuality" of these objects (Jay, 1988)—the glass is witnessed at arm's length, whereas the lake is witnessed at throwing distance. Accordingly, the overall perturbation looms larger in the case of the glass (a splash) than in the case of the lake (just a ripple): In the case of the glass, the perturbation is depicted as a storm in a teacup—a commotion within a confined space, an upheaval in the form of vertical displacement of water far exceeding the diameter of the containing vessel. This same local impact, however, is experienced in the case of the lake as but a drop in the ocean—the lake's vast expanse mitigates the impact. To wit, whereas the lake is in fact several orders of magnitude larger than the glass, its perspectival embodiment is gestured as roughly equivalent (see Figure 3). Moreover, the initial impact of the marble on the lake surface would ironically be far greater than in the glass, because the marble would have hit the lake surface at a far greater velocity. Thus, Li regains his maximal grip on the world—he reengages traction on the histogram by grounding its anomalous behavior in embodied proto-proportional gestalts.

Pragmatics of analogy

In selecting a baseball context for his second analogy, Li evokes a situation that is more conducive to mathematizing his argument. This analogy is much nearer to the probability context—semantically, cultural–semiotically, and arithmetically—than the water-surface

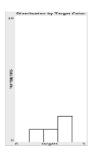
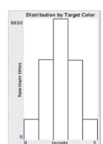
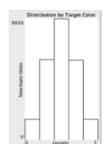
Context	**Before**	**Perturb** →	**Effect:** Major impact	**Before**	**Perturb** →	**Effect**: Barely discernable impact
Marbles-Scooping Experiment		+1 sample			+1 sample	
	↕		↕	↕		↕
Marble in Water	Cup	throw marble	"Splash"	Lake	throw marble	"Ripple"
				↕		↕
Batting Average	–	–	–	Batting average at 500 at-bats (x/500)	One out	Batting average at 501 at-bats (x/501)

FIGURE 3 Analogizing the proportional impact of an addend as a function of the size of the aggregate. Adding a single sample to a small outcome distribution is like throwing a marble into a glass—you get a splash. Adding a single sample to a large outcome distribution is like throwing a marble into a lake—you get just a ripple.

analogy. Li thus steers the dialogue beginning from the rich imagistic detour and through progressive mathematization back to the institutionally normative genre of discussing quantitative reasoning numerically.

Using numbers as precise signifiers of quantity, in the baseball analogy, Li need not communicate magnitudes gesturally—indeed, comparison of the sweeping "big giant lake" gesture to the modest "500 at-bats" cupping gesture as well as, analogically, comparison of the lake-impact gesture to the "go down like two points" gesture demonstrate pragmatic contraction in iconicity from the first to the second context, a contraction typically marking assumed interpersonal coherence (see Radford, 2008).

Having completed his water and baseball argumentation sequence, Li loops the dyad back to the phenomenon under scrutiny—"Look, they're just barely moving." Presumably, the interlocutors, who renew their joint attention to the histogram, do so with the newly shared professional vision established in the analogical discursive space. As an aside, perhaps the most interesting aspect of this conversation between Li and the interviewer is in the unsaid—it is the fact that Dor understood Li immediately, even though Li's multimodal marble metaphor was structurally complex yet multimodally and rapidly delivered. As Núñez, Edwards, and Matos (1999) propose, such situated intersubjectivity is possible due to the shared biology of the interlocutors' embodied minds.

Tradeoffs in the design of instructional representations

Embedding sophisticated features into a mathematical artifact may introduce interaction advantages of great utility to an expert. However, when this artifact is used as a pedagogical object, the learner needs to unpack the embedded features in order to use the artifact meaningfully. We thus implicate the technologically sophisticated self-adjusting histogram as complicit to enhancing the tension between the tacit and cultural presentation of quantitative information. Specifically, by self-adjusting the heights of its columns proportionally, the electronic histogram's elegant functionality trades off spatial fidelity for pragmatic representation. Li's imagistic analogical detour suggests that he needed to step away from the sophisticated device to a space where he could construct an imagistic bridge, which in turn enabled him to numerically objectify his intuitive comparison. Ultimately, the entire image lemma served as a discursive means to orient Dor's view of the histogram.

Implicit to the process of mathematization is expressing multimodal images in conventional inscription. This meta-representational task is cognitively nontrivial (diSessa & Sherin, 2000). Pedagogical responses to this general design problem, we believe, are in fostering opportunities for students to customize the negotiation of phenomenological qualia and their cultural reconstructions (Abrahamson, 2009a, 2009c, 2009d; Clement, 1993). Idiosyncratic metaphor appears to be one such means.

Comparison of the Three Data Excerpts

We have presented and analyzed three cases in which individuals in an interviewer–student dyad generated mathematical metaphors to express their idiosyncratic perceptual construction of situations under joint scrutiny. The artifacts ranged in their mathematical complexity, and thus the metaphors ranged both in their structural complexity and the requisite elaborateness of their intersubjective rendition, from few words and gestures toward the situations to brief narrative detours that temporarily shifted the interlocutors' co-attention away from the situations into an alternative spatial–temporal discursive space.

In all three cases, the dyads assumed metaphorical reasoning as a means of redeploying the negotiation of mathematical meaning away from contested formal procedures involving imposed semiotic media over to shared informal experiential contexts wherein intuitive inference can be applied, articulated, and co-validated. To achieve this discursive shift, the dyads temporarily relaxed socio-mathematical norms for what constitutes institutionally sanctioned genres and topics (Gomes & Meira, 2006; Walkerdine, 1988). So doing, the dyads implicitly agreed to suspend, problematize, and query the asymmetrical social capital and hierarchical positioning inherent to prevalent discourse of mathematics. Namely, counter to prevalent institutional convention, the dyads endorsed tacit perceptual judgment and inference mechanisms as the epistemological grounds of disciplinary practice (Polanyi, 1967, p. 21; Varela, 1999, p. 7). In particular, instead of rehearsing algorithms, the dyads searched for extra-mathematical vantage points from which unfamiliar mathematical views could be "intimated" (Sfard, 2002) as meaningful. Thus metaphor enables and forms intersubjective ways of seeing things, of being on the same page.

GENERAL SUMMARY, CONCLUSIONS, AND IMPLICATIONS

We have furnished several examples of a student and an interviewer–researcher who conjured elaborate metaphors apparently as a means of helping the student make sense of an unfamiliar mathematical artifact or manage the performance of a challenging mathematical solution procedure. We suggested that these metaphors enlist embodied experiences from everyday phenomenology and, so doing, co-opt existing and well-rehearsed mental schemes; these schemes help learners generate relevant inferences and better manage cognitive resources. The empirical examples demonstrated both the promise and fragility of spontaneous idiosyncratic metaphor as a discursive support. More generally, we submit that educational research could ultimately be more useful to theory building and to practice if researchers more explicitly emphasized and examined analogies inherent in students' thinking. We end this article with several conclusions and implications for the theory and practice of mathematics education.

Roles of Metaphor in Learning, Teaching, and Educational Research on Mathematics

Metaphors that students and instructors generate apparently spontaneously, as they engage in learning and teaching mathematical subject matter, are interesting to study, because they appear to bear unique affordances for students, teachers, and researchers.

For students, metaphors are a powerful discursive technique of building meaning while making sense of novel situations, including unfamiliar mathematical artifacts.

For teachers, metaphors indicate the processes, and not just the products, of students' attempts to understand new ideas; metaphors reveal aspects of students' reasoning that might otherwise remain implicit.

For researchers, metaphors appear to occupy a middle space between, on the one hand, students' situated, informal, imagistic personal experiences and, on the other hand, their deliberate goal-oriented actions with mathematical objects. By closely observing students as they elicit, leverage, and metaphorize their personal experiences as means of accomplishing their mathematical investigations, we can gain insight into an apparent epistemological tension between tacit and formal knowledge and, so doing, we can bring into question some implicit assumptions about the nature of tacit resources as they relate to mathematical knowledge.

In particular, we echo other researchers and philosophers who argue that experience is far richer than mathematical knowledge, and that in making sense of mathematical objects students unpack their experience and look at it in a new way. Specifically, the root of mathematical meaning is in tacit knowledge and informal experiences; students reason analogically so as to operate in familiar territory where they can make sense of mathematical artifacts, draw inferences, and build arguments. Yet in order to be able to use their experiences analogically, students may need to unpack the tacit work of their inaccessible cognitive mechanisms, sometimes in surprising ways. For example, proportion is built into the perceptual mechanism as a tacitly perspectival experience, so that proportional comparisons can become flattened into direct comparisons of simulated retinal magnitude. Yet leveraging tacit mechanisms as instruments of mathematical reasoning requires a supportive environment. Thus, teachers should be sensitive to students' analogical struggles and be equipped to nurture and guide it. We are thus suggesting pragmatic connections between the anthropology of instruction (Goodwin, 1994) and of meaning (Radford, 2006), with an emphasis on discourse and mathematical artifacts (Sfard, 2007). That is, people accomplish conceptual progress by trying to show others how they see things.

Given the explorative nature of this study, these conclusions and the following implications should be regarded as emerging hypotheses in need of further investigation. With that caveat in mind, this article makes the following further contributions to research on cognition and instruction.

Implications for Theory of Learning

Metaphor can accomplish significant work for mathematics students and teachers. We have proposed and demonstrated the plausibility that metaphor, similar to symbols, gesture, and diagrams, should be theorized as a semiotic means of objectification and thus should be added to the roster of semiotic means discussed in the semiotic–cultural literature as enabling mathematical learning, reasoning, and communicating. Moreover, metaphor should be recognized as a unique means. Although metaphors are ultimately expressed linguistically, in speech, gesture, and in conjunction with a variety of symbolic displays—all being modalities and semiotic systems that have already been previously implicated in the literature as offering means of objectification—metaphors reveal an important function in students' learning that cuts across these modalities. Namely, generative analogical reasoning both affords powerful bridging between situations and their purported models and facilitates the application of new strategies.

By focusing on spontaneous metaphorical constructions, we are in a sense looking not at contexts that designers parcel a priori into instructional activities students engage but at idiosyncratic contexts that students bring to bear associatively in making ad hoc sense of situations. It is precisely the idiosyncratic, improvisatory nature of these discursive contributions, we conjecture, that make metaphorical reasoning powerful conduits for individuals to negotiate tacit and mathematical meaning of situations.

Implications for Mathematics Pedagogy

When we think, we always think about some thing in some way. When that thing is new, we try to make sense of it by figuring out what it is like, how we should see or use it. As we begin to make sense of new things, it can be difficult to articulate to ourselves or explain to others how we are seeing the things, moreover when the new things are complicated. For example, mathematical and techno–scientific phenomena, including situations, instruments, and symbolic displays, as well as normative solution procedures presented by an instructor as appropriate for these phenomena, are often difficult to first make sense of and communicate, because understanding the phenomena's disciplinary meanings is contingent on objectifying and interpreting their implicit inner properties, relations, aspects, patterns, structure, dynamics, rationale, function, etc. Consequently, we sometimes take advantage of apparently analogous situations that the new things evoke, and we use metaphor to express these analogies as our way of seeing the things.

In general, mathematics students resort to metaphorical discourse when they evaluate the formal register either as inaccessible, insufficient, or otherwise inappropriate to express their emerging logico–quantitative construction of phenomena. By communicating their views explicitly in metaphorical propositions, students create opportunities to reflect on their own reasoning more concretely as well as to receive customized guidance from a teacher who is willing to see things their way. Similarly, teachers shift formal discourse to metaphorical space when doing so appears better to enable student insight.

At the same time, informal reasoning has been described as a double-edged sword (Cobb, 1989), and metaphor is no exception. In particular, idiosyncratic fanciful constructions can be unruly agents in instructional discourse, because they tend to demand more dedicated attention and individualized scaffolding than either traditional classroom pedagogy or sheer time limitations are likely to afford (Rick, 2009).

Nevertheless, the constructivist philosophy of learning suggests that idiosyncratic metaphor is precisely the form of discursive contribution that educational frameworks should accommodate, because students draw on diverse epistemic and cognitive resources (Abrahamson, Berland, Shapiro, Unterman, & Wilensky, 2006; Makar & Rubin, 2009; Turkle & Papert, 1991). Metaphor should thus be embraced and positioned as a socio-mathematical normative discourse genre. Licensing metaphors as valuable discursive contributions may empower teachers to enable a more diverse body of students to leverage their cognitive funds as means of grounding mathematical concepts (see diSessa, in Abrahamson, 2006).

Particularly for probabilistic notions involving chance events that are difficult to objectify, metaphor appears to support students' sense-making by bridging situations and concepts, that is, bridging the "epistemological distance" (Morgan, Mariotti, & Maffei, 2009), "rupture" (Radford, 2003), or "cognitive gap" (Herscovics & Linchevski, 1996) between tacit phenomenology of situations and their explicit cultural reformulation (see also Bamberger & diSessa, 2003).

Operating in the design-based research approach, we have been developing and investigating pedagogical frameworks for engineering bidirectional bridges across the tacit–cultural interface. That is, we have been creating and researching instructional solutions that support students' personal construction of bridges between unmediated views of problematized situations and mediated analytical reformulations of these situations (Abrahamson, 2009b; Abrahamson & Wilensky, 2007).

ACKNOWLEDGEMENTS

This article draws on empirical data collected in a study supported by a National Academy of Education/Spencer Postdoctoral Fellowship 2005–2006 (*Seeing Chance,* Abrahamson). A synopsis of the paper focusing on different data was presented at SRTL-6 (Abrahamson, 2009d). An elaborated treatment of the Li episode has appeared in Abrahamson (2010). We are grateful to all members of the Embodied Design Research Laboratory at the University of California, Berkeley (Abrahamson, *Director*), for generative contention during the microgenetic analysis of our data. Thanks to Katie Makar, Dani Ben-Zvi, and the MTL editors and anonymous reviewers of earlier drafts who helped us realize how we see things. The second and third authors, a graduate student and an undergraduate research apprentice, respectively, made equivalent contributions to this article—their names are listed in reverse-alphabetical order.

REFERENCES

Abrahamson, D. (2004). Embodied spatial articulation: A gesture perspective on student negotiation between kinesthetic schemas and epistemic forms in learning mathematics. In D. E. McDougall & J. A. Ross (Eds.), *Proceedings of the Twenty Sixth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 791–797). Toronto, Ontario, Canada: Preney.

Abrahamson, D. (2006). What's a situation in situated cognition?—A constructionist critique of authentic inquiry (Symposium). In S. Barab, K. Hay, & D. Hickey (Eds.), *Proceedings of the Seventh International Conference of the Learning Sciences* (Vol. 2, pp. 1015–1021). Bloomington, IN: ICLS.

Abrahamson, D. (2009a). Embodied design: Constructing means for constructing meaning. *Educational Studies in Mathematics*, *70*(1), 27–47.

Abrahamson, D. (2009b). Orchestrating semiotic leaps from tacit to cultural quantitative reasoning—the case of anticipating experimental outcomes of a quasi-binomial random generator. *Cognition and Instruction*, *27*(3), 175–224.

Abrahamson, D. (2009c). A student's synthesis of tacit and mathematical knowledge as a researcher's lens on bridging learning theory. *International Electronic Journal of Mathematics Education*, *4*(3), 195–226. Retrieved from http://www.iejme.com/032009/main.htm.

Abrahamson, D. (2009d). Coordinating phenomenologically immediate and semiotically mediated constructions of statistical distribution. In K. Makar (Ed.), *The role of context and evidence in informal inferential reasoning. Proceedings of the Sixth International Research Forum on Statistical Reasoning, Thinking, and Literacy (SRTL-6)*. Brisbane, Australia: The University of Queensland.

Abrahamson, D. (2010). A tempest in a teapot is but a drop in the ocean: Action-objects in analogical mathematical reasoning. In K. Gomez, L. Lyons, & J. Radinsky (Eds.), *Learning in the disciplines: Proceedings of the 9th International Conference of the Learning Sciences (ICLS 2010)* (Vol. 1 [Full Papers], pp. 492–499. Chicago, IL: International Society of the Learning Sciences.

Abrahamson, D., & Cendak, R. M. (2006). The odds of understanding the Law of Large Numbers: A design for grounding intuitive probability in combinatorial analysis. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the Thirtieth Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 1–8. Charles University, Prague, Czech Republic: PME.

Abrahamson, D., & Wilensky, U. (2007). Learning axes and bridging tools in a technology-based design for statistics. *International Journal of Computers for Mathematical Learning*, *12*(1), 23–55.

Abrahamson, D., Janusz, R., & Wilensky, U. (2006). There once was a 9-Block . . .—A middle-school design for probability and statistics [Electronic Version]. *Journal of Statistics Education, 14*(1). Retrieved from http://www.amstat.org/publications/jse/v14n1/abrahamson.html.

Abrahamson, D., Berland, M. W., Shapiro, R. B., Unterman, J. W., & Wilensky, U. (2006). Leveraging epistemological diversity through computer-based argumentation in the domain of probability. *For the Learning of Mathematics*, *26*(3), 39–45.

Abrahamson, D., Bryant, M. J., Gutierrez, J. F., Mookerjee, A. V., Souchkova, D., & Thacker, I. E. (2009). Figuring it out: Mathematical learning as guided semiotic disambiguation of useful yet initially entangled intuitions. In S. L. Swars, D. W. Stinson, & S. Lemons-Smith (Eds.), *Proceedings of the Thirty-First Annual Meeting of the North-American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 662–670). Atlanta, GA: Georgia State University.

Arcavi, A. (1994). Symbol sense: Informal sense-making in formal mathematics. *For the Learning of Mathematics*, *14*(3), 24–35.

Artigue, M., Cerulli, M., Haspekian, M., & Maracci, M. (2009). Connecting and integrating theoretical frames: The TELMA contribution. *International Journal of Computers for Mathematical Learning*, *14*, 217–240.

Badets, A., & Pesenti, M. (2010). Creating number semantics through finger movement perception. *Cognition*, *115*(1), 46–53.

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal*, *93*(4), 373–397.

Bamberger, J. (1999). Action knowledge and symbolic knowledge: The computer as mediator. In D. Schön, B. Sanyal, & W. Mitchell (Eds.), *High technology and low Income communities* (pp. 235–262). Cambridge, MA: MIT Press.

Bamberger, J., & diSessa, A.A. (2003). Music as embodied mathematics: A study of a mutually informing affinity. *International Journal of Computers for Mathematical Learning*, *8*(2), 123–160.

Barsalou, L. W. (2008). Grounded cognition. *Annual Review of Psychology*, *59*, 617–645.

Bartolini Bussi, M. G., & Boni, M. (2003). Instruments for semiotic mediation in primary school classrooms. *For the Learning of Mathematics*, *23*(2), 12–19.

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artefacts and signs after a Vygotskian perspective. In L. D. English, M. G. Bartolini Bussi, G. A. Jones, R. Lesh, & D. Tirosh (Eds.), *Handbook of international research in mathematics education*, 2nd revised edition (pp. 720–749). Mahwah, NJ: Lawrence Erlbaum.

Batanero, C., Navarro-Pelayo, V., & Godino, J. D. (1997). Effect of implicitly combinatorial model on combinatorial reasoning in secondary school pupils. *Educational Studies in Mathematics*, *32*, 181–199.

Benfey, O. T. (1958). August Kekulé and the birth of the structural theory of organic chemistry in 1858. *Journal of Chemical Education*, *35*, 21–23.

Black, M. (1993). More about metaphor. In A. Ortony (Ed.), *Metaphors and thought* (pp. 19–41). Cambridge, UK: Cambridge University Press.

Case, R., & Okamoto, Y. (Eds.). (1996). *The role of central conceptual structures in the development of children's thought* (Vol. 61[1–2], Serial No. 246). Chicago, IL: University of Chicago Press.

Clement, J. (1993). Using bridging analogies and anchoring intuitions to deal with students' preconceptions in physics. *Journal of Research in Science Teaching*, *30*(10), 1241–1257.

Clement, J., Brown, B., & Zietsman, A. (1989). Not all preconceptions are misconceptions: Finding "anchoring conceptions" for grounding instruction on students' intuitions. *International Journal of Science Education*, *11*(5), 554–565.

Cobb, P. (1989). A double-edged sword [Review of the book Intuition in science and mathematics]. *Journal for Research in Mathematics Education*, *20*(2), 213–218.

Cobb, P., & Bauersfeld, H. (Eds.). (1995). *The emergence of mathematical meaning: Interaction in classroom cultures*. Hillsdale, NJ: Lawrence Erlbaum.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, *32*(1), 9–13.

Dewey, J. (1928/2008). Preoccupation with the disconnected In J. A. Boydston (Ed.), *The collected works of John Dewey: Later works* Volume3: 1927–1928 *—essays, reviews, miscellany* (pp. 25–40). Carbondale and Edwardsville, IL: Southern Illinois University Press. (From "Body and Mind." First published in the Bulletin of the NY Academy of Medicine, 1928).

diSessa, A. A. (1983). Phenomenology and the evolution of intuition. In D. Gentner & A. Stevens (Eds.), *Mental models* (pp. 15–33). Hillsdale, NJ: Lawrence Erlbaum.
diSessa, A. A. (1993). Towards an epistemology of physics. *Cognition and Instruction*, *10*(2–3), 105–225.
diSessa, A. A. (2007). An interactional analysis of clinical interviewing. *Cognition and Instruction*, *25*(4), 523–565.
diSessa, A. A. (2008). A note from the editor. *Cognition and Instruction*, *26*(4), 427–429.
diSessa, A. A., & Sherin, B. (2000). Meta-representation: An introduction. *Journal of Mathematical Behavior*, *19*, 385–398.
diSessa, A. A., & Cobb, P. (2004). Ontological innovation and the role of theory in design experiments. *The Journal of the Learning Sciences*, *13*(1), 77–103.
diSessa, A. A., Philip, T. M., Saxe, G. B., Cole, M., & Cobb, P. (2010, April–May). *Dialectical approaches to cognition (Symposium)*. Paper presented at the Annual Meeting of American Educational Research Association, Denver, CO.
Dove, G. (2009). Beyond perceptual symbols: A call for representational pluralism. *Cognition*, *110*(3), 412–431.
Edwards, L., Radford, L., & Arzarello, F. (2009). Gestures and multimodality in the construction of mathematical meaning [Special Issue]. *Educational Studies in Mathematics*, *70*(2).
English, L. D. (2005). Combinatorics and the development of children's combinatorial reasoning. In G. A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning* (pp. 121–141). Dordrecht, The Netherlands: Kluwer.
Fauconnier, G., & Turner, M. (2002). *The way we think: Conceptual blending and the mind's hidden complexities*. New York, NY: Basic Books.
Fillmore, C. J. (1976). Frame semantics and the nature of language. *Annals of the New York Academy of Sciences, 280,* 20–32.
Fodor, J. A. (1980). Fixation of belief and concept acquisition. In M. Piattelli-Palmerini (Ed.), *Language and learning: The debate between Jean Piaget and Noam Chomsky* (pp. 142–149). Cambridge, MA: Harvard University Press.
Gentner, D. (2001). Spatial metaphors in temporal reasoning. In M. Gattis (Ed.), *Spatial schemas and abstract thought* pp. 203–222. Cambridge, MA: MIT. Press.
Ginsburg, H. P. (1997). *Entering the child's mind*. New York, NY: Cambridge University Press.
Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Chicago, IL: Aldine.
Glenberg, A. M. (1997). What memory is for. *Behavioral and Brain Sciences*, *20*, 1–55.
Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517–545). Mahwah, NJ: Lawrence Erlbaum.
Gomes, G., & Meira, L. (2006). The discourse of logical necessity: Rules for action in pre-school mathematics. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the Thirtieth Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 201–208). Charles University, Prague, Czech Republic: PME.
Goodwin, C. (1994). Professional vision. *American Anthropologist*, *96*(3), 603–633.
Groth, R. E., & Bergner, J. A. (2005). Preservice elementary school teachers' metaphors for the concept of statistical sample. *Statistics Education Research Journal*, *4*(2), 27–42.
Hadamard, J. (1945). *The psychology of invention in the mathematical field*. New York, NY: Dover.
Herscovics, N., & Linchevski, L. (1996). Crossing the cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics*, *30*(2), 39–65.
Hutchins, E. (2005). Material anchors for conceptual blends. *Journal of Pragmatics*, *37*(10), 1555–1577.
Jay, M. (1988). Scopic regimes of modernity. In H. Foster (Ed.), *Vision and visuality* (pp. 3–23). New York, NY: New Press.
Johnson, M. L. (1987). *The body in the mind: The bodily basis of meaning, imagination, and reason*. Chicago, IL: Chicago University Press.
Kelly, A. E. (2004). Design research in education: Yes, but is it methodological? *Journal of the Learning Sciences*, *13*(1), 115–128.
Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York, NY: Basic Books.

Lemke, J. L. (1998). Multiplying meaning: Visual and verbal semiotics in scientific text. In J. R. Martin & R. Veel (Eds.), *Reading science: Critical and functional perspectives on discourses of science* (pp. 87–113). London, UK: Routledge.

Makar, K., & Rubin, A. (2009). A framework for thinking about informal statistical inference. *Statistics Education Research Journal*, *8*(1), 82–105.

Mariotti, M. A. (2009). Artifacts and signs after a Vygotskian perspective: The role of the teacher. *ZDM: The International Journal on Mathematics Education*, *41*, 427–440.

Merleau-Ponty, M. (1958/2005). *Phenomenology of perception* (C. Smith, trans.). New York, NY: Routledge. (Original work published 1945.)

Morgan, C., Mariotti, M. A., & Maffei, L. (2009). Representation in computational environments: epistemological and social distance. *International Journal of Computers for Mathematical Learning*, *14*, 241–263.

Nemirovsky, R. (2003). Three conjectures concerning the relationship between body activity and understanding mathematics. In R. Nemirovsky & M. Borba (Coordinators), Perceptuo-motor activity and imagination in mathematics learning (Research Forum). In N. A. Pateman, B. J. Dougherty & J. T. Zilliox (Eds.), *Twenty Seventh Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 105–109). Honolulu, HI; Columbus, OH: Eric Clearinghouse for Science, Mathematics, and Environmental Education.

Nemirovsky, R., & Borba, M. C. (2004). PME special issue: Bodily activity and imagination in mathematics learning. *Educational Studies in Mathematics*, *57*, 303–321.

Newman, D., Griffin, P., & Cole, M. (1989). *The construction zone: Working for cognitive change in school*. New York, NY: Cambridge University Press.

Núñez, R. E., Edwards, L. D., & Matos, J. F. (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational Studies in Mathematics*, *39*, 45–65.

Papert, S. (1980). *Mindstorms: Children, computers, and powerful ideas*. New York, NY: Basic Books.

Piaget, J. (1968). *Genetic epistemology* (E. Duckworth, trans.). New York, NY: Columbia University Press.

Pirie, S., & Kieren, T. E. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics*, *26*, 165–190.

Poincaré, J. H. (1897/2003). *Science and method* (F. Maitland, trans.). New York, NY: Dover.

Polanyi, M. (1967). *The tacit dimension*. London, UK: Routledge & Kegan Paul.

Presmeg, N. C. (1986). Visualisation in high school mathematics. *For the Learning of Mathematics*, *6*(3), 42–46.

Presmeg, N. C. (1992). Prototypes, metaphors, metonymies and imaginative rationality in high school mathematics. *Educational Studies in Mathematics*, *23*(6), 595–610.

Presmeg, N. C. (1998). Metaphoric and metonymic signification in mathematics. *Journal of Mathematical Behavior*, *17*(1), 25–32.

Presmeg, N. C. (2006). Research on visualization in learning and teaching mathematics: Emergence from psychology. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present, and future* (pp. 205–235). Rotterdam, The Netherlands: Sense.

Puntambekar, S., & Sandoval, W. A. (2009). Design research: Moving forward (Editor's note). *The Journal of the Learning Sciences*, *18*(3), 323–326.

Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning*, *5*(1), 37–70.

Radford, L. (2006). The anthropology of meaning. *Educational Studies in Mathematics*, *61*(1/2), 39–65.

Radford, L. (2008). Iconicity and contraction: A semiotic investigation of forms of algebraic generalizations of patterns in different contexts. *ZDM: The International Journal on Mathematics Education*, *40*(1), 83–96.

Resnick, L. B. (1992). From protoquantities to operators: Building mathematical competence on a foundation of everyday knowledge. In G. Leinhardt, R. Putnam, & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 373–429). Hillsdale, NJ: Lawrence Erlbaum.

Rick, W. (2009). *Metaphors & analogies: Power tools for teaching any subject*. Portland, ME: Stenhouse.

Roediger, H. L., & McDermott, K. B. (1993). Implicit memory in normal human subjects. In F. Boller & J. Grafman (Eds.), *Handbook of Neuropsychology* (Vol. 8, pp. 63–131). Amsterdam. The Netherlands: Elsevier.

Roth, W.-M. (2001). Situating cognition. *Journal of the Learning Sciences*, *10*(1/2), 27–61.

Rotman, B. (2000). *Mathematics as sign: Writing, imagining, counting*. Stanford, CA: Stanford University Press.

Rowland, T. (1999). Pronouns in mathematics talk: Power, vagueness and generalisation. *For the Learning of Mathematics*, *19*(2), 19–26.

Rubin, A., & Hammerman, J. (2007). Soup or stew: Metaphors for the relationship between samples and populations. In D. Pratt & J. Ainsley (Eds.), *Reasoning about informal inferential statistical reasoning: A collection of current research studies – Proceedings of the 5th International Research Forum on Statistical Reasoning, Thinking, and Literacy (SRTL5).* Warwick, UK: University of Warwick.

Ruthven, K., Laborde, C., Leach, J., & Tiberghien, A. (2009). Design tools in didactical research: Instrumenting the epistemological and cognitive aspects of the design of teaching sequences. *Educational Researcher*, *38*(5), 329–342.

Schegloff, E. A. (1996). Confirming allusions: Toward an empirical account of action. *The American Journal of Sociology*, *102*(1), 161–216.

Schoenfeld, A. H. (1991). On mathematics as sense-making: An informal attack on the unfortunate divorce of formal and informal mathematics. In J. F. Voss, D. N. Perkins, & J. Segal (Eds.), *Informal reasoning and education* (pp. vii–xvii). Hillsdale, NJ: Lawrence Erlbaum.

Schoenfeld, A. H. (2004). The math wars. *Educational Policy*, *18*, 253–286.

Schoenfeld, A. H., Smith, J. P., & Arcavi, A. (1991). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology* (pp. 55–175). Hillsdale, NJ: Lawrence Erlbaum .

Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics*, *14*(1), 44–55.

Sfard, A. (2002). The interplay of intimations and implementations: Generating new discourse with new symbolic tools. *Journal of the Learning Sciences*, *11*(2/3), 319–357.

Sfard, A. (2007). When the rules of discourse change, but nobody tells you—making sense of mathematics learning from a commognitive standpoint. *Journal of Learning Sciences*, *16*(4), 567–615.

Siegler, R. S., & Crowley, K. (1991). The microgenetic method: A direct means for studying cognitive development. *American Psychologist*, *46*(6), 606–620.

Sriraman, B., & English, L. D. (2004). Combinatorial mathematics: Research into practice. *Mathematics Teacher*, *98*, 182–191.

Steiner, G. (2001). *Grammars of creation*. New Haven, CT: Yale University Press.

Stevens, R., & Hall, R. (1998). Disciplined perception: Learning to see in technoscience. In M. Lampert & M. L. Blunk (Eds.), *Talking mathematics in school: Studies of teaching and learning* (pp. 107–149). New York, NY: Cambridge University Press.

Thagard, P. (2010). How brains make mental models. In L. Magnani, W. Carnielli, & C. Pizzi (Eds.), *Model-based reasoning in science and technology: Abduction, logic, and computational discovery* (pp. 447–461). Berlin, Germany: Springer.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics*, *3*, 165–208.

Turkle, S., & Papert, S. (1991). Epistemological pluralism and the revaluation of the concrete. In I. Harel & S. Papert (Eds.), *Constructionism* (pp. 161–192). Norwood, NJ: Ablex.

van den Heuvel-Panhuizen, M. (2003). The didactical use of models in Realistic Mathematics Education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics*, *54*(1), 9–35.

Varela, F. J. (1999). *Ethical know-how: Action, wisdom, and cognition*. Stanford, CA: Stanford University Press.

Vygotsky, L. S. (1934/1962). *Thought and language*. Cambridge, MA: MIT Press.

Walkerdine, V. (1988). *The mastery of reason: Cognitive development and the production of rationality*. London, UK: Routledge.

Wilensky, U. (1997). What is normal anyway?: Therapy for epistemological anxiety. *Educational Studies in Mathematics*, *33*(2), 171–202.

Wilensky, U. (1999). *NetLogo*. Northwestern University, Evanston, IL: The Center for Connected Learning and Computer-Based Modeling. Retrieved from http://ccl.northwestern.edu/netlogo/.

Wilson, M. (2002). Six views of embodied cognition. *Psychonomic Bulletin & Review*, *9*(4), 625–636.